The Simplex Method

In Chapters 3 and 4 we introduced linear programming and showed how models with two variables can be solved graphically. We relied on computer programs (WINQSB, Excel, or LINDO) to generate optimal solutions and sensitivity analyses for problems with more than two variables. The algorithm used in each of these programs is a variant of the simplex method, first developed by George Dantzig in 1947. In recent years, much attention has been paid to a new technique known as an *interior point algorithm*. While this algorithm has proven successful for solving many types of large problems, the simplex method remains the predominant method for solving most problems. Here we present a review of what's going on in the simplex computer module of software packages.

1 Standard and Canonical Form

STANDARD FORM

The **simplex method** for solving linear programming models requires that *all functional constraints be written as equalities* so that *elementary row operations*—(1) multiplying equations by positive or negative numbers and (2) adding multiples of one equation to other equations—can systematically be performed without changing the set of feasible solutions to the problem. The problem must also be expressed in nonnegative variables. When these conditions are met, the problem is said to be in **standard form**.

Standard Form

A linear program is in *standard form* if all the functional constraints are written as equations and all the variables are required to be nonnegative.

Converting a Linear Programming Formulation to Standard Form

When a linear programming model is formulated, it will most likely include some inequality constraints and perhaps some variables that are not restricted to be nonnegative. To achieve standard form, the following conversions are made:

1. *"≤" Constraints*—Define a nonnegative *slack variable* (SI) to represent the difference between the left side and the right side of the constraint. Then add SI to the right side of the constraint to form an equation.

   Example: $2X1 + 5X2 - 3X3 \leq 120$

   Define $S1$ = the difference between 120 and $2X1 + 5X2 - 3X3$

   Result: $2X1 + 5X2 - 3X3 + S1 = 120$

2. *"≥" Constraints*—Define a nonnegative *surplus variable* (SI) to represent the difference between the right side and the left side of the constraint. Then subtract SI from the right side of the constraint to form an equation.

Example: $6X1 - 5X2 + 4X3 \geq 200$

Define S1 = the difference between $6X1 - 5X2 + 4X3$ and 200

Result: $6X1 - 5X2 + 4X3 - S1 = 200$

3. XJ *is restricted to a nonpositive variable*—Define a new nonnegative variable $XJ' = -XJ$ and replace XJ by $-XJ'$ in the formulation.

4. XJ *is unrestricted* (i.e., it can be positive, negative, or 0)—Define $XJ = XJ' - XJ''$ where XJ' and XJ'' are restricted to nonnegative variables and replace XJ in the formulation by $XJ' - XJ''$.

Using these rules, let us convert the following linear programming formulation to one in standard form. Note that the value of I that we use for a slack or surplus variable corresponds to the position of the constraint in the formulation; that is, S1 is associated with the first constraint, S2 with the second, and so on.

$$
\begin{array}{llr}
\text{MAX} & 2X1 + 5X2 - 4X3 + 8X4 & \\
\text{ST} & X1 + X2 + 3X3 - 2X4 \geq & 28 \\
 & 6X1 + 5X2 + X4 = & 30 \\
 & 7X1 - 2X2 + 4X3 \leq & 25 \\
 & X2 - 3X3 \geq & 1 \\
 & X1, X4 \geq 0,\ X2 \text{ unrestricted},\ X3 \leq 0 &
\end{array}
$$

Now we define:

S1 = Surplus variable for constraint 1

S3 = Slack variable for constraint 3

S4 = Surplus variable for constraint 4

$X2 = X2' - X2''$

$X3 = -X3'$

Making these substitutions, the standard form is:

$$
\begin{array}{llr}
\text{MAX} & 2X1 + 5X2' - 5X2'' + 4X3' + 8X4 & \\
\text{ST} & X1 + X2' - X2'' - 3X3' - 2X4 - S1 = & 28 \\
 & 6X1 + 5X2' - 5X2'' + X4 = & 30 \\
 & 7X1 - 2X2' + 2X2'' - 4X3' + S3 = & 25 \\
 & X2' - X2'' + 3X3' - S4 = & 1 \\
 & X1, X2', X2'', X3', X4, S1, S3, S4 \geq 0 &
\end{array}
$$

CANONICAL FORM

After slack and surplus variables have been added to the functional constraints, there are typically more total variables (decision, slack, and surplus variables) than there are equations. When this occurs, there are usually an infinite number of possible solutions. Even so, it may be difficult to determine even one solution. For example, can you easily find a solution to the following three equations in six unknowns?

$$
\begin{aligned}
6X1 + X2 + 2X3 + 5X4 + X5 + 9X6 &= 100 \\
12X1 + 3X2 + 4X3 + 9X4 + X5 + 23X6 &= 170 \\
3X1 + X2 + X3 + 7X4 + X5 + 7X6 &= 80
\end{aligned}
$$

There is no immediately obvious solution to this system of equations. However, if a system of equations is written in **canonical form**, as defined below, a solution can easily be determined:

Canonical Form

A system of equations is in *canonical form* if for *each* equation there exists a variable that appears only in that equation, and its coefficient in that equation is +1.

When we perform a series of elementary row operations on a system of equations, the result is an *equivalent system* of equations that has exactly the same set of solutions as the original system of equations. By performing an appropriate series of elementary row operations on the above system of three equations in six unknowns, we can show the following is an equivalent system of equations:

$$\begin{aligned} 3X1 \quad + X3 - 2X4 \quad\quad + 2X6 &= 20 \\ X2 \quad\quad + 4X4 \quad\quad + 5X6 &= 15 \\ 5X4 + X5 \quad\quad &= 45 \end{aligned}$$

This system is in canonical form because X3 appears only in the first constraint and has a coefficient +1 in this equation; X2 appears only in the second constraint with a coefficient of +1; and X5 appears only in the third constraint also with a coefficient of +1. These variables (X3, X2, and X5) are the *basic variables* corresponding to these equations. The other variables (X1, X4, and X6) are called *nonbasic variables*. Note that X1 also appears only in the first equation. Since its coefficient is +3, not +1, however it is not the basic variable for the first equation.[1]

Basic and Nonbasic Variables

Each equation in a system of equations in canonical form has a *basic variable* which is multiplied by +1 in that equation and multiplied by 0 (does not appear) in the other equations.

Variables that are not basic variables are *nonbasic* variables.

When a system of equations is written in canonical form, we find the solution by setting *all* the nonbasic variables to 0; then, the values of the basic variables are the corresponding numbers on the right-hand side of the equations. This is called a **basic solution**. Thus, for the problem above, we can obtain the values of the basic variables by setting X1 = 0, X4 = 0, and X6 = 0; then X3 = 20, X2 = 15, and X5 = 45, respectively. The basic solution corresponding to the system of equations written in the above canonical form is, therefore:

$$X1 = 0,\ X2 = 15,\ X3 = 20,\ X4 = 0,\ X5 = 45,\ X6 = 0$$

Since this system of equations is equivalent to the original set of equations, the above solution is also a feasible solution for the original set of equations. We can verify this by substituting these values into the original set of equations:

$$\begin{aligned} 6(0) + 1(15) + 2(20) + 5(0) + 1(45) + 9(0) &= 100 \\ 12(0) + 3(15) + 4(20) + 9(0) + 1(45) + 23(0) &= 170 \\ 3(0) + 1(15) + 1(20) + 7(0) + 1(45) + 7(0) &= 80 \end{aligned}$$

[1] Had the coefficient of X1 in the first equation also been +1, instead of +3, then either X1 or X3 could have been selected as the basic variable for the first equation. The selection is arbitrary; the variable not selected is designated as a nonbasic variable.

In linear programming models, a feasible solution must satisfy not only the functional constraints, but also the nonnegativity constraints of the variables; that is, all variables must be greater than or equal to zero. A basic solution in which all the variables have nonnegative values is a **basic feasible solution** for the problem. In the basic solution above, all the variables do have nonnegative values; thus, it is a basic feasible solution.

Basic Solutions and Basic Feasible Solutions

A *basic solution* for a system of equations in canonical form is obtained by setting the nonbasic variables to zero and the basic variables to the right-hand side values.

A basic solution in which all the variables are greater than or equal to zero is a *basic feasible solution*.

Basic feasible solutions are important because of the following algebraic/geometric property for feasible regions generated by linear constraints:

Basic Feasible Solution/Extreme Point Equivalence

A basic feasible solution is equivalent to an extreme point of the feasible region of a linear constraint set, and vice versa.

Recall that, according to the extreme point property, if a linear program has an optimal solution, then an extreme point must be optimal. Thus, algebraically, if a linear program has an optimal solution, then a basic feasible solution must be optimal.

II Tableaus for Maximization Problems When All Functional Constraints Are "≤" Constraints

The simplex method performs elementary row operations on functional constraints written in canonical form. In the discussion that follows, we outline the simplex method for problems with a maximization objective function and all "≤" functional constraints. (Problems with other structures are discussed later in this supplement.) We illustrate the approach by considering the model for Galaxy Industries presented in Chapter 3 of the text. For this problem, the linear programming model is:

X1 = dozens of Space Rays produced in the production run

X2 = dozens of Zappers produced in the production run

$$
\begin{array}{llll}
\text{MAX} & 8X1 + 5X2 & & \text{(Profit)} \\
\text{ST} & & & \\
& 2X1 + X2 \leq & 1200 & \text{(Plastic)} \\
& 3X1 + 4X2 \leq & 2400 & \text{(Production Time)} \\
& X1 + X2 \leq & 800 & \text{(Total Production)} \\
& X1 - X2 \leq & 450 & \text{(Mix)} \\
& X1, X2 \geq 0 & &
\end{array}
$$

We define:

S1 = the amount of unused plastic
S2 = the amount of unused production time
S3 = the amount by which total production falls below 800 dozen
S4 = the amount by which the difference in production of Space Rays and Zappers falls below 450 dozen

The standard form for the Galaxy Industries model is then:

$$
\begin{array}{lrcl}
\text{MAX} & 8X1 + 5X2 & & \text{(Profit)} \\
\text{ST} & & & \\
 & 2X1 + X2 + S1 & = 1200 & \text{(Plastic)} \\
 & 3X1 + 4X2 \quad + S2 & = 2400 & \text{(Production Time)} \\
 & X1 + X2 \qquad + S3 & = 800 & \text{(Total Production)} \\
 & X1 - X2 \qquad\qquad + S4 & = 450 & \text{(Mix)} \\
 & \text{All } XJ \geq 0 \text{ and all } SJ \geq 0 & &
\end{array}
$$

As you can see, since the slack variables S1, S2, S3, and S4 comprise a set of basic variables, this standard form is also a canonical form. In fact, canonical form always results from adding slack variables to a model in which all the functional constraints are "≤" constraints.

The initial basic feasible solution, found by setting the nonbasic variables X1 and X2 to 0, is X1 = 0, X2 = 0, S1 = 1200, S2 = 2400, S3 = 800, and S4 = 450. Since X1 = 0, X2 = 0 yields an objective function value of zero, hopefully this is not the *optimal* solution. The simplex method operates by performing elementary row operations on this set of equations to generate an equivalent set of equations that is also in canonical form. It then determines if the basic feasible solution for this equivalent set of equations is optimal; if it is not, it repeats the process.

One efficient way to keep track of the equations and other relevant information utilized in the simplex method is through the use of a matrix (similar to a spreadsheet) called the *simplex tableau*. Different software packages format the simplex tableau differently. Here we present one similar to that used by WINQSB. Instructions for constructing a simplex tableau are illustrated in Figure CD3.1.

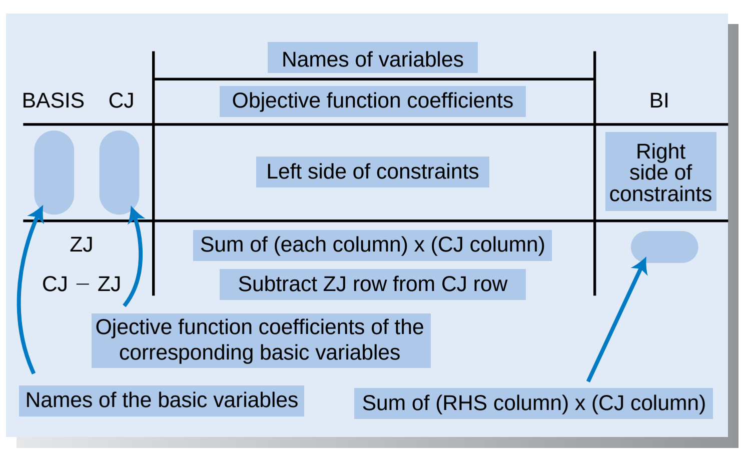

FIGURE CD3.1 Constructing a Tableau

The simplex tableau for the initial canonical form for the Galaxy Industries problem is given in Figure CD3.2. Let us analyze the information contained in this tableau.

BASIS	CJ	X1	X2	S1	S2	S3	S4	BI
		8	5	0	0	0	0	
S1	0	2	1	1	0	0	0	1200
S2	0	3	4	0	1	0	0	2400
S3	0	1	1	0	0	1	0	800
S4	0	1	−1	0	0	0	1	450
ZJ		0	0	0	0	0	0	0
CJ − ZJ		8	5	0	0	0	0	

FIGURE CD3.2 Initial Tableau for the Galaxy Industries Problem

THE BASIS, CJ, AND BI COLUMNS

The BASIS column of the simplex tableau keeps track of the basic variables. The CJ column provides the original objective function coefficients for the corresponding variables listed in the BASIS column. The BI column gives the right-hand values of the constraints written in their current form. Since the basic variables are set to the values on the right side of the equations, written in canonical form, the BI column also gives the values of the corresponding basic variables listed in the BASIS column.

THE ROWS OF THE TABLEAU

With the exception of the ZJ and CJ-ZJ rows, the rows of the tableau are fairly self-explanatory. The CJ row provides the original objective function coefficients for all the variables. The rows in the body of the tableau give the coefficients of the equations, written in their current canonical form.

The CJ Row

The CJ row gives the objective function coefficients of all the variables, including slack variables. If a variable is increased by one unit, its CJ value represents the *gross increase* in the value of the objective function. In the original tableau for Galaxy Industries, if X1 is increased by one unit, the gross increase in the value of the objective function is 8, as designated by the entry of 8 in its CJ row. Similarly, if X2 is increased by one unit, the gross increase in the value of the objective function is 5, etc.

The ZJ Row

If a variable is increased from its current value, the values of other variables must also change so that each equation is satisfied. Changing the value of variables can change the overall contribution those variables make to the objective function, however. In maximization problems, the ZJ row gives the per unit *gross decrease* to the value of the objective function due to these changes. The entries in this row are obtained by summing the products of the entries in its variable column and the corresponding entries in the CJ column.

For example, consider the equations in the above tableau:

$$
\begin{aligned}
2X1 + X2 + S1 \quad &= 1200\\
3X1 + 4X2 + S2 \quad &= 2400\\
X1 + X2 + S3 \quad &= 800\\
X1 - X2 + S4 &= 450
\end{aligned}
$$

The corresponding basic feasible solution is X1 = 0, X2 = 0, S1 = 1200, S2 = 2400, S3 = 800, S4 = 450.

Let us now determine the effect on the constraints if X1 is allowed to increase from 0 while all other nonbasic variables (in this case, only X2) remain 0. Keeping X2 = 0, we see that if X1 = 1, S1 has to be 1198 in order for the first equation to hold; S2 has to be 2397 for the second equation to hold; S3 has to be 799 for the third equation to hold; and S4 has to be 449 for the fourth equation to hold. In other words, as X1 increases by one unit, S1 decreases by two, S2 decreases by three, S3 decreases by one, and S4 decreases by one. These are the numbers found in the X1 column. Thus, we can interpret the numbers in the main body of the tableau as follows:

Interpretation of Numbers in the Body of the Tableau

The numbers in the column of a nonbasic variable provide the amount the corresponding basic variables will *decrease* given a one-unit increase in that nonbasic variable.

Hypothetical ZJ Calculation Now, suppose, hypothetically, that the objective function coefficients for S1, S2, S3, and S4 were 2, 1, 0, and 3, respectively (*they are not but suppose they were*). In this case, the two-unit decrease in S1 reduces the objective function value by 2(2) = \$4; the three-unit decrease in S2 reduces the objective function by 3(1) = \$3; the one-unit decrease in S3 decreases it by 1(0) = \$0; and the one-unit decrease in S4 decreases it by 1(3) = \$3. The total gross decrease in profit is \$4 + \$3 + \$0 + \$3 = \$10; this becomes the ZJ entry in the X1 column. Table CD3.1 summarizes the calculations.

TABLE CD3.1 (HYPOTHETICAL) CALCULATION OF THE ZJ VALUE FOR XI

Basic Variable	Amount Decreased (XI Column)	(Hypothetical) Profit Loss per Unit (CJ Column)	Total Profit Loss (XI × CJ)
S1	2	\$2	\$4
S2	3	\$1	\$3
S3	1	\$0	\$0
S4	1	\$3	\$3
		Total per Unit Decrease (ZJ Value) =	\$10

Actual ZJ calculation for the original tableau The actual objective function coefficients for S1, S2, S3, and S4 in the first tableau are not 2, 1, 0, and 3, respectively, but 0, 0, 0, and 0. Thus, the actual ZJ calculation for X1 in the first tableau is shown in Table CD3.2. These data verify the ZJ entry for X1 of 0 in the original tableau. To calculate the ZJ value for another variable, we would simply perform the same calculation using the column of that variable in place of the X1 column. In this tableau, there is no gross decrease in the value of the objective function for any variable (the ZJs are all 0).

TABLE CD3.2 (ACTUAL) CALCULATION OF THE ZJ VALUE FOR XI

Basic Variable	Amount Decreased (XI Column)	Profit Lost per Unit (CJ Column)	Total Lost Profit (XI × CJ)
S1	2	\$0	\$0
S2	3	\$0	\$0
S3	1	\$0	\$0
S4	1	\$0	\$0
		Total per Unit Decrease (ZJ Value) =	\$0

The entry in the BI column of the ZJ row is obtained the same way as are the other ZJ values. Since the CJ column gives the objective function coefficients of the basic variables and the BI column gives the value of the basic variables, this entry gives the *value of the objective function* for the basic feasible solution associated with this tableau. Because the basic feasible solution for this first tableau shows X1 = 0 and X2 = 0, the objective function value for this solution is 0.

The CJ-ZJ Row

The CJ row provides the gross *per unit increase* in the value of the objective function when each variable is increased, and the ZJ row gives the corresponding gross *per unit decrease* in the value of the objective function when each variable is increased. The difference between the per unit gross increase and the per unit gross decrease is the *net effect* on the value of the objective function as each variable is increased by one unit. The net effect on the objective function of increasing X1 by one unit is $8; for X2 it is $5; and so on. Note that the CJ-ZJ *value for all basic variables is* 0.

Meaning of CJ, ZJ, and CJ-ZJ Rows for Maximization Problems

CJ Value — The *gross increase* in the value of the objective function, given a one-unit increase in that variable.

ZJ Value — The *gross decrease* in the value of the objective function given a one-unit increase in that variable.

CJ-ZJ Value — The *net effect* on the value of the objective function, given a one-unit increase in that variable.

(For minimization problems, the CJ value gives the *gross decrease* and the ZJ value gives the *gross increase* to the value of the objective function, given a one-unit increase in the variable. The CJ-ZJ value still gives the *net effect* of the change.)

Figure CD3.3 summarizes the meaning of each number in a simplex tableau.

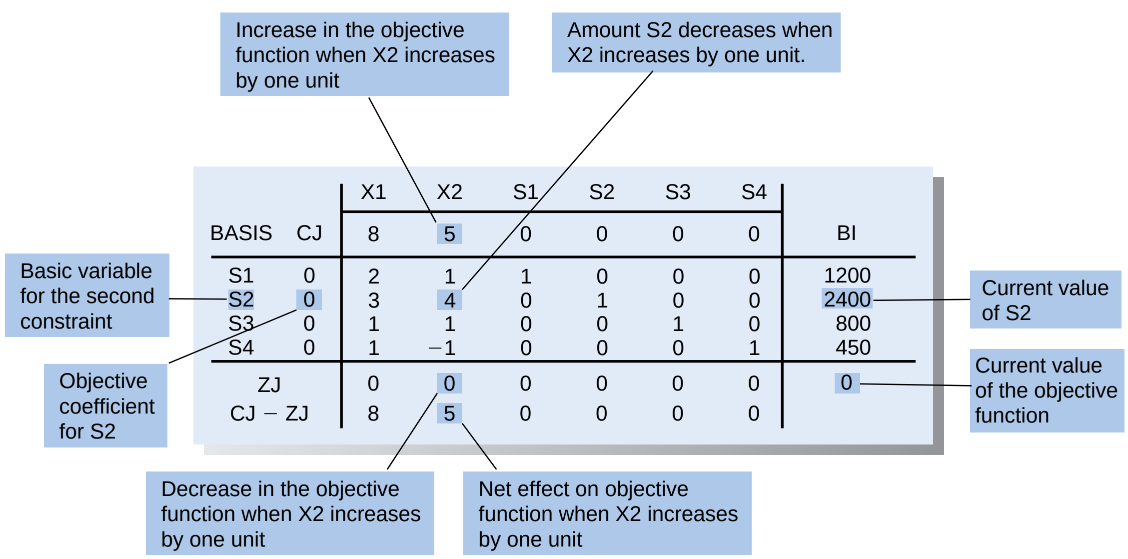

FIGURE CD3.3 Meaning of all Entries in a Tableau

III The Simplex Algorithm

The simplex method is a three-step procedure involving the following concepts:

The Simplex Algorithm: The Approach

1. Evaluate whether or not the current basic feasible solution is optimal. If it is not, determine the nonbasic variable that can be increased from zero which will improve the value of the objective function at the fastest rate.
2. Determine how much this variable can be increased before a current basic variable is forced to zero.
3. Perform elementary row operations to generate an equivalent system in which the nonbasic variable found in Step 1 replaces the basic variable identified in Step 2 as a basic variable.

Return to Step 1

We now illustrate this approach by analyzing the original tableau for the Galaxy Industries developed in section II.

STEP 1: DETERMINE IF THE CURRENT SOLUTION IS OPTIMAL (MAXIMIZATION PROBLEMS)

You will recall that the CJ-ZJ row gives the net effect of a one-unit increase in each variable on the objective function. If there are no positive entries in this row, an increase in any variable will not improve the value of the objective function, and the current solution is optimal.

Since there are positive CJ-ZJ values of 8 and 5 for X1 and X2, respectively, this indicates that improvement is possible by increasing either of these variables. The objective function will increase $8 for every unit X1 is increased and $5 for every unit X2 is increased. We select X1, the variable with the *most positive* CJ-ZJ value, as the current nonbasic variable that will increase from 0. It is called the *entering variable* since it will *enter* the set of basic variables at the next iteration. The column of the entering variable is called the *entering*, or *pivot column*.

Thus, the rule for selecting the entering variable in Step 1 can be expressed as follows:

Step 1: Which Variable Increases the Objective Function Value the Most per Unit?

Select the variable with the most positive entry in the CJ-ZJ row as the entering variable. If there are no positive entries in the CJ-ZJ row, STOP. The current solution is optimal.

We pointed out that increasing either X1 or X2 (variables with positive CJ-ZJ values) increases the objective function value. We selected X1 because it increases the objective function the most per unit. If we had chosen to increase X2 instead, we would still eventually reach the optimal solution; by selecting the variable with the most positive CJ-ZJ value, however, the optimal solution is usually (although not always) reached more quickly.[2] If two or more variables have the same largest positive CJ-ZJ value, the entering variable can be selected arbitrarily from them.

STEP 2: DETERMINE THE MAXIMUM INCREASE FOR THE ENTERING VARIABLE

As we have seen, if X1 increases by one unit, S1 decreases by two units. Thus, if X1 increases by 30 units, S1 decreases by 30(2) = 60 units to 1200 − 60 = 1140; if X1 increases by 100 units, S1 decreases by (100)(2) = 200 units to 1200 − 200 = 1000. Since the value of S1 is currently 1200, the ratio 1200/2 = 600 is the maximum increase in X1 possible before S1 reaches 0. If X1 increases by more than 600, S1 becomes negative, which is not permitted. Similar reasoning indicates the following:

Equation	***Ratio***	***Interpretation***
1	1200/2 = 600	Maximum value of X1 before S1 reaches 0
2	2400/3 = 800	Maximum value of X1 before S2 reaches 0
3	800/1 = 800	Maximum value of X1 before S3 reaches 0
4	450/1 = 450	Maximum value of X1 before S4 reaches 0

Since no variable, including S1, S2, S3, or S4 can have a negative value, the *maximum* value that X1 can take on at this time, before some basic variable reaches zero, is the *minimum ratio* of 450. When X1 = 450, the current basic variable, S4, is the first basic variable to reach zero. This variable is called the *leaving variable* because it will *leave* the set of basic variables at the next iteration. The current row corresponding to the leaving variable is called the *leaving*, or *pivot row*.

In this case, there are no negative or zero entries in the X1 column. If there had been a negative entry inthe X1 column in a particular row, the corresponding basic variable would *increase* (not decrease) as X1 increases. Similarly, if there had been a zero entry in the X1 column in a particular row, the corresponding basic variable would have remained unchanged (rather than decrease) as X1 increases. Neither of these cases imposes limits on the maximum value of X1. Hence, *this ratio test is applied only to positive numbers in the entering column*.

We can express the process for selecting the leaving variable in Step 2 as follows:

Step 2: How Far Can the Entering Variable Be Increased?

Find the minimizing ratio between the right-hand side values and *positive* entries in the entering column.

STEP 3: GENERATE AN EQUIVALENT SYSTEM OF EQUATIONS WITH A NEW BASIS REPRESENTATION

Since only basic variables can be positive (nonbasic variables are set to zero), we need a new canonical form representation for the equations in which the entering variable replaces the leaving variable as a basic variable. The goal is to make the new column of the entering variable (the pivot column) all zeros, except for a +1 in the row corresponding to the row of the leaving variable (the pivot row), as shown in Figure CD3.4.

The current value of the entry that we wish to make +1 is the number that is in both the pivot row and pivot column; it is called the *pivot element*. Thus, at this stage, we wish to generate a new tableau in which all of the entries in the X1 column (the pivot column) are zeros, except for a +1 in the fourth row (since S4 is the leaving variable). To accomplish this, we employ the elementary row operations in the following sequence.

[2]This concept is illustrated in Problem 4 at the end of this supplement.

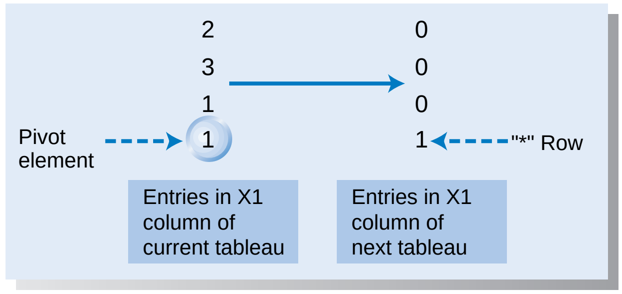

FIGURE CD3.4 Goal of Elementary Row Operations

1. Generate a "+1" in the pivot row

It just so happens that in this first tableau, there is already a +1 entry for X1 in the pivot row; that is, the pivot element is a 1. But if the pivot element had been a "5," say, dividing the pivot row through by 5 would generate a row with a +1 entry in the X1 column. Thus, to "get the 1" in the pivot row, we *divide the pivot row by the pivot element*. We designate the new row generated as a result of this division as the "*" row, in a new tableau as shown in Figure CD3.5.

		X1	X2	S1	S2	S3	S4		
BASIS	CJ	8	5	0	0	0	0	BI	Ratio
S1	0	2	1	1	0	0	0	1200	600
S2	0	3	4	0	1	0	0	2400	800
S3	0	1	1	0	0	1	0	800	800
S4	0	1	−1	0	0	0	1	450	450 ← Pivot row
ZJ		0	0	0	0	0	0	0	0
CJ − ZJ		8	5	0	0	0	0		

Generate the "*"row

Get a "1" in the position of the pivot element
by dividing the pivot row by the pivot element

		X1	X2	S1	S2	S3	S4		
BASIS	CJ	8	5	0	0	0	0	BI	
		1	−1	0	0	0	1	450	(*)
ZJ									
CJ − ZJ									

FIGURE CD3.5 Generating the "*" Row

2. Generate the "0s" in the other rows

Consider the first equation row. If there is already a "0" in that row, we simply copy the row into the new tableau. Since the coefficient in the first row is a "2," however, we can make this coefficient a "0" in the new tableau by another elementary row operation. Recall that an equivalent system of equations is generated if a multiple of one row is added to or subtracted from another row. Since there is now a "1" in the pivot column of the "*" row in the new tableau, we can generate a "0" in the pivot column for the first equation by subtracting 2 times the "*" row from the current row. This is illustrated in Figure CD3.6.

		X1	X2	S1	S2	S3	S4		
BASIS	CJ	8	5	0	0	0	0	BI	Ratio
S1	0	2	1	1	0	0	0	1200	600
S2	0	3	4	0	1	0	0	2400	800
S3	0	1	1	0	0	1	0	800	800
S4	0	1	−1	0	0	0	1	450	450 ← Pivot row
ZJ		0	0	0	0	0	0	0	0
CJ − ZJ		8	5	0	0	0	0		

1. Write the current first row :	2	1	1	0	0	0	1200
2. Multiply "*" row by 2 :	2	−2	0	0	0	1	900
3. Subtract (1) − (2) :	0	3	1	0	0	−2	300

		X1	X2	S1	S2	S3	S4		
BASIS	CJ	8	5	0	0	0	0	BI	
		0	3	1	0	0	−2	300	
		1	−1	0	0	0	1	450	(*)
ZJ									
CJ − ZJ									

FIGURE CD3.6 Generating the "0" in the First Equation

Similarly, subtracting 3 times the "*" row from the current second row yields a "0" in the X1 column of the second row of the next tableau; subtracting 1 times the "*" row from the current third row yields a "0" in the third row of the X1 column of the next tableau. Once these steps are completed, we are left with an equivalent set of equations, with X1 replacing S4 as a basic variable.

These steps for generating the equivalent set of equations for the next tableau can be summarized as follows:

Step 3: How Are the Equations for the Next Tableau Generated?

1. For *pivot row*: Divide pivot row by pivot element to get the "*" row for the next tableau.
2. For *other rows*: Multiply this "*" row by the value in the current pivot column and subtract the result from the current row.

3. Complete the Tableau

We can now fill in the rest of the next tableau. The BASIS entries remain the same, except that X1 replaces S4 as a basic variable and its corresponding BASIS CJ column entry is 8. The ZJ entries are calculated in the same manner as before by summing the multiples of the entries in the CJ column by the corresponding entries in each column and the CJ-ZJ entries are found by subtracting the ZJ row from the CJ row. For example, the ZJ entry for X2 is now $0(3) + (0)(7) + (0)(2) + (8)(-1) = -8$, and its CJ-ZJ value is now $5 - (-8) = 13$.

The complete tableau resulting from these row operations is shown in Figure CD3.7.

		X1	X2	S1	S2	S3	S4		
BASIS	CJ	8	5	0	0	0	0	BI	Ratio
S1	0	2	1	1	0	0	0	1200	600
S2	0	3	4	0	1	0	0	2400	800
S3	0	1	1	0	0	1	0	800	800
S4	0	1	−1	0	0	0	1	450	450 ← Pivot row
ZJ		0	0	0	0	0	0	0	
CJ − ZJ		8	5	0	0	0	0		

1. Write the current second row :	3	4	0	1	0	0	2400
2. Multiply "*" row by 3 :	3	−3	0	0	0	3	1350
3. Subtract (1) − (2) : Goal is to get "0" here	0	7	0	1	0	−3	1050

1. Write the current third row :	1	1	0	0	1	0	800
2. Multiply "*" row by 1 :	1	−1	0	0	0	1	450
3. Subtract (1) − (2) : Goal is to get "0" here	0	2	0	0	0	−1	350

		X1	X2	S1	S2	S3	S4	
BASIS	CJ	8	5	0	0	0	0	BI
S1	0	0	3	1	0	0	−2	300
S2	0	0	7	0	1	0	−3	1050
S3	0	0	2	0	0	1	−1	350
X1	8	1	−1	0	0	0	1	450
ZJ		8	−8	0	0	0	8	3600
CJ − ZJ		0	13	0	0	0	−8	

FIGURE CD3.7 Generating the "0" in the Second and Third Equations

The set of steps in the simplex algorithm can now be summarized as follows:

Simplex Algorithm (For Maximization Problems): The Mechanics

1. Find the most positive entry in the CJ-ZJ row; this is the entering variable. If there are no positive entries, STOP; the current solution is optimal.
2. Find the minimizing ratio between the RHS values and positive entries in the entering column; this is the leaving variable.
3. **a.** Divide pivot row by pivot element to get the "*" row for the next tableau.
   b. For *other rows*: Multiply this "*" row by the current pivot column value and subtract the result from the current row.
   c. Replace the leaving variable by the entering variable in the BASIS column and enter its CJ coefficient in the BASIS CJ column. Calculate the ZJ value for each column by summing the products of the BASIS CJ entries and the corresponding entry in the column. Calculate the CJ-ZJ row by subtracting the ZJ row entries from the CJ row entries.

Then go back to Step 1

Each repetition of these steps is an *iteration* of the algorithm. The tableau we generated above by replacing S4 with X1 as a basic variable is the tableau for the second iteration. Note that, at this point, the solution is now X1 = 450, X2 (nonbasic) = 0, S1 = 300, S2 = 1050, S3 = 350, S4 (nonbasic) = 0, yielding an objective function value of 3600.

SUBSEQUENT ITERATIONS

We will do one additional iteration for the Galaxy Industries problem in detail. In Figure CD3.7, the tableau for the second iteration, the most positive CJ-ZJ value is 13. This corresponds to the X2 column; hence, X2 is the entering variable. The ratios for Step 2 are determined by dividing the right-hand side values by the numbers in the X2 column. These ratios for the first three rows are 300/3 = 100, 1050/7 = 150, and 350/2 = 175, respectively. No ratio is determined for the fourth row because its value in the pivot column is a negative number (−1). Since the minimizing ratio is 100, S1 is the leaving variable and the first row is the "*" row in the next tableau. The pivot element, found at the intersection of the entering column and leaving row, is 3.

To generate the "*" row for the next tableau, we divide the first row of the current tableau by the pivot element, (3). The resulting iteration is illustrated in Figure CD3.8.

BASIS	CJ	X1	X2	S1	S2	S3	S4	BI	Ratio
		8	5	0	0	0	0		
S1	0	0	3	1	0	0	−2	300	100 (min)
S2	0	0	7	0	1	0	−3	1050	150
S3	0	0	2	0	0	1	−1	350	175
X1	0	1	−1	0	0	0	1	450	- - -
ZJ		8	−8	0	0	0	8	3600	
CJ − ZJ		0	13	0	0	0	−8		

Divide pivot row (first row) by pivot element (3) to get "*" row

BASIS	CJ	X1	X2	S1	S2	S3	S4	BI	
		8	5	0	0	0	0		
		0	1	1/3	0	0	−2/3	100	(*)
ZJ									
CJ − ZJ									

FIGURE CD3.8 Iteration 2: Generating The "*" Row

The remainder of the equations are found by multiplying this "*" row by the current entry in the X2 column for each row and subtracting the result from the current row entries. Thus, the new second row is generated by subtracting 7 times the "*" row from the current second row; the new third row by subtracting 2 times the "*" row from the current third row; and the new fourth row by subtracting −1 times the "*" row from the current fourth row (or alternatively, *adding* +1 times the "*" row to the current fourth row). Then, X2 replaces S1 in the BASIS column, its CJ value of 5 is entered next to it in the BASIS CJ column, and the ZJ row and the CJ-ZJ rows are calculated.

These calculations are shown in detail in Figure CD3.9, the tableau to begin the next iteration. The basic feasible solution associated with this tableau is X1 = 550, X2 = 100, S1 (nonbasic) = 0, S2 = 350, S3 = 150, and S4 (nonbasic) = 0. The objective function value is 4900.

To do the third iteration, we note that the most positive CJ-ZJ value in Figure CD3.9 is $\frac{2}{3}$; therefore, S4 is the entering variable. We determine the ratios for Step 2 by dividing the right-hand side values by the corresponding positive values in the S4 column; no ratio is calculated for the first row since its pivot column element ($-\frac{2}{3}$) is negative. The steps for this iteration are shown in Figure CD3.10.

(1) Write the current second row :	0	7	0	1	0	−3	|	1050
(2) Multiply "*" row by 7 :	0	∓	7/3	0	0	−14/3	|	700
(3) New second row: (1) − (2) :	0	0	−7/3	1	0	5/3	|	350

Goal is to get "0" here

(1) Write the current third row :	0	2	0	0	1	−1	|	350
(2) Multiply "*" row by 2 :	0	2	2/3	0	0	−4/3	|	200
(3) New third row: (1) − (2) :	0	0	−2/3	0	1	1/3	|	150

Goal is to get "0" here

(1) Write the current fourth row :	1	−1	0	0	0	1	|	450
(2) Multiply "*" row by −1 :	1	−1	−1/3	0	0	2/3	|	−100
(3) New fourth row: (1) − (2) :	1	0	1/3	0	0	1/3	|	550

Goal is to get "0" here

Gives the tableau for the third iteration

BASIS	CJ	X1 8	X2 5	S1 0	S2 0	S3 0	S4 0	BI
X2	5	0	1	1/3	0	0	−2/3	100
S2	0	0	0	−8/3	1	0	5/3	350
S3	0	0	0	−2/3	0	1	1/3	150
X1	8	1	0	1/3	0	0	1/3	550
ZJ		8	0	−13/3	0	0	−2/3	4900
CJ − ZJ		0	0	−13/3	0	0	2/3	

FIGURE CD3.9 Iteration 2: Calculations for Rows 2, 3, and 4

BASIS	CJ	X1 8	X2 5	S1 0	S2 0	S3 0	S4 0	BI	Ratio
X2	5	0	1	1/3	0	0	−2/3	100	- - -
S2	0	0	0	−8/3	1	0	5/3	350	210 (min)
S3	0	0	0	−2/3	0	1	1/3	150	450
X1	8	1	0	1/3	0	0	1/3	550	1650
ZJ		8	0	−13/3	0	0	−2/3	4900	
CJ − ZJ		0	0	−13/3	0	0	2/3		

1. Divide pivot row by 5/3 get "*" row.
2. Subtract (−2/3) X ("*" row) from first row.
3. Subtract (1/3) X ("*" row) from third row.
4. Subtract (1/3) X ("*" row) from fourth row.

This gives :

BASIS	CJ	X1 8	X2 5	S1 0	S2 0	S3 0	S4 0	BI
X2	5	0	1	−3/5	2/5	0	0	240
S4	0	0	0	−7/5	3/5	0	1	210 (*)
S3	0	0	0	−1/5	−1/5	1	0	80
X1	8	1	0	4/5	−1/5	0	0	480
ZJ		8	5	17/5	2/5	0	0	5040
CJ − ZJ		0	0	−17/5	−2/5	0	0	

FIGURE CD3.10 The Third Iteration

The basic feasible solution associated with this tableau is X2 = 240, S4 = 210, S3 = 80, X1 = 480, and the nonbasic variables, S1 = 0, and S2 = 0. The value of the objective function is 5040. Since there are no positive CJ-ZJ values in Figure CD3.10, the algorithm is terminated and we conclude that the above solution is optimal. Note that the result, X1 = 480, X2 = 240, coincides with the result we derived graphically in Chapter 3.

IV Geometric Interpretation of the Simplex Algorithm

The simplex algorithm evaluates an extreme point (basic feasible solution) and determines if it is optimal. If it is determined that the solution is not optimal, then the algorithm evaluates an adjacent extreme point (basic feasible solution) that provides a better value for the objective function and determines if that point is optimal. This process continues until an optimal extreme point (basic feasible solution) is found. The extreme points generated by the simplex method for the Galaxy Industries problem in the previous section were:

Iteration 1:	X1 = 0, X2 = 0	Objective Function:	0
Iteration 2:	X1 = 450, X2 = 0	Objective Function:	3600
Iteration 3:	X1 = 550, X2 = 100	Objective Function:	4900
Iteration 4:	X1 = 480, X2 = 240	Objective Function:	5040

As indicated in Figure CD3.11, these iterations generate a set of adjacent extreme points.

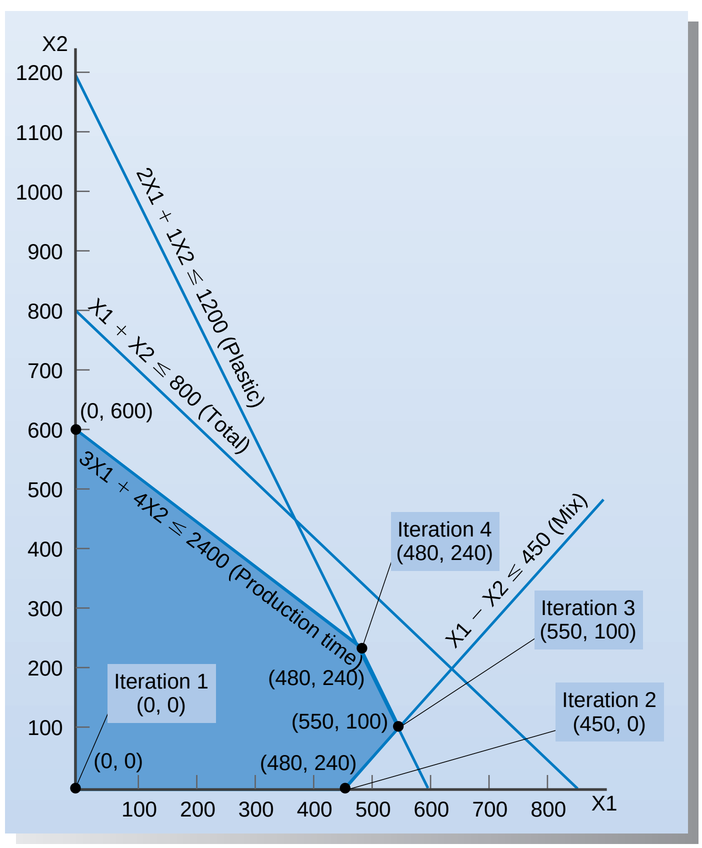

FIGURE CD3.11 Sequence of Extreme Points Generated by the Simplex Algorithm for the Galaxy Industries Problem

V The simplex method when some functional constraints are not "≤" constraints

Suppose we wish to solve the following linear programming model:

$$
\begin{array}{lrrrrcr}
\text{MAX} & 16X1 + & 15X2 + & 20X3 - & 18X4 & & \\
\text{ST} & 2X1 + & X2 + & 3X3 & & \leq & 3000 \\
 & 3X1 + & 4X2 + & 5X3 - & 60X4 & \leq & 2400 \\
 & & & & X4 & \leq & 32 \\
 & & X2 & & & \geq & 200 \\
 & X1 + & X2 + & X3 & & \geq & 800 \\
 & X1 - & X2 - & X3 & & = & 0 \\
 & \multicolumn{6}{c}{XJ \geq 0 \text{ for all } J}
\end{array}
$$

To convert all constraints to equalities, we must first add slack variables S1, S2, and S3 to the first three "≤" constraints and subtract surplus variables S4 and S5 from the left side of the fourth and fifth constraints, respectively. The sixth constraint is already an equation; thus, it does not require the addition of any variables. The result of these operations is the following standard form for this model:

$$
\begin{array}{lrrrrrrrrrcr}
\text{MAX} & 16X1 + & 15X2 + & 20X3 - & 18X4 & & & & & & & \\
\text{ST} & 2X1 + & X2 + & 3X3 & & + S1 & & & & & = & 3000 \\
 & 3X1 + & 4X2 + & 5X3 - & 60X4 & & + S2 & & & & = & 2400 \\
 & & & & X4 & & & + S3 & & & = & 32 \\
 & & X2 & & & & & & - S4 & & = & 200 \\
 & X1 + & X2 + & X3 & & & & & & - S5 & = & 800 \\
 & X1 - & X2 - & X3 & & & & & & & = & 0 \\
 & \multicolumn{11}{c}{XJ \geq 0,\ SJ \geq 0 \text{ for all } J}
\end{array}
$$

As you can see, however, this standard form is *not* in canonical form. While the S1, S2, and S3 columns have the appropriate structure for the first, second, and third functional constraints, the other three constraints do not include a variable that appears only in that constraint and has a coefficient of "+1." A "−1" coefficient, such as S4 and S5, in the fourth and fifth constraints, does not meet this requirement.

Since the simplex algorithm requires canonical form to perform its operations, we can employ a mathematical "trick" to jumpstart the problem. We add *artificial variables* (AI's) to the equations that do not currently have a basic variable. Thus we add artificials A4, A5, A6, respectively, to the fourth, fifth, and sixth equations. These artificial variables, along with S1, S2, and S3, then serve as the first set of basic variables for the simplex method. Hence, *artificial variables are added to constraints that were initially "≥" constraints or "=" constraints.*

These artificial variables do not really exist, however, and *as long as any artificial variable is positive, the corresponding solution is not really a feasible solution* for the original problem. Thus, in addition to maximizing the objective function, we must also drive all the artificial variables to zero.

These two goals can be merged into one by assigning a very large negative objective function coefficient, −M, (+M for minimization problems) to each artificial variable.[3] The idea is that, since M is a very large number (close to infinity), if the value of some corresponding artificial variable, AJ, is even the least bit positive, its contribution to the objective function, −M(AJ), is such a large negative number that it is effectively considered negative infinity.

[3] An alternative method that treats the two goals separately is illustrated in Problem 9 at the end of this supplement.

Since the simplex algorithm seeks to maximize the value of the objective function, it attempts to improve the solution from negative infinity. The only way to do this is to set each artificial variable equal to zero. Thus, the simplex algorithm automatically seeks to reduce the artificial variables to zero as it attempts to improve the objective function from one iteration to the next:

Adding Artificial Variables to Obtain An Initial Canonical Form

1. *Artificial variables* are added to each functional constraint formulated as a "≥" or an "=" constraint.
2. Each artificial variable is assigned a coefficient of −M (+M in minimization problems) in the objective function.

The following is the initial canonical form for the model after artificial variables have been added to the above standard form:

$$\begin{array}{ll}
\text{MAX} & 16X1 + 15X2 + 20X3 - 18X4 - MA4 - MA5 - MA6 \\
\text{ST} & 2X1 + X2 + 3X3 + S1 = 3000 \\
 & 3X1 + 4X2 + 5X3 - 60X4 + S2 = 2400 \\
 & X4 + S3 = 32 \\
 & X2 - S4 + A4 = 200 \\
 & X1 + X2 + X3 - S5 + A5 = 800 \\
 & X1 - X2 - X3 + A6 = 0 \\
 & XJ \geq 0,\ SJ \geq 0,\ AJ \geq 0 \text{ for all } J
\end{array}$$

The simplex algorithm can then be used to solve this problem.

SOLVING FOR THE OPTIMAL SOLUTION WHEN THERE ARE ARTIFICIAL VARIABLES

The tableaus for even the small problem given above are quite large. To demonstrate how to solve for an optimal solution when there are artificial variables and to relate the graphical approach, we instead consider the following two-decision variable problem:

Formulation

$$\begin{array}{ll}
\text{MAX} & 2X1 + 5X2 \\
\text{ST} & X1 \geq 4 \\
 & X1 + 4X2 \leq 32 \\
 & 3X1 + 2X2 = 24 \\
 & X1, X2 \geq 0
\end{array}$$

By adding the appropriate slack, surplus, and artificial variables, we obtain the following canonical form (note that the numbering of the slack, surplus, and artificial variables corresponds to the number of the functional constraint in which they are placed):

Canonical Form

$$\begin{array}{ll}
\text{MAX} & 2X1 + 5X2 - MA1 - MA3 \\
\text{ST} & X1 - S1 + A1 = 4 \\
 & X1 + 4X2 + S2 = 32 \\
 & 3X1 + 2X2 + A3 = 24 \\
 & X1, X2, S1, S2, A1, A3 \geq 0
\end{array}$$

Figures CD3.12*a–d* are the series of tableaus generated by the simplex algorithm to reach the optimal solution. In these tableaus, since M is a very large number, the variable with the largest positive coefficient for M in the CJ-ZJ row is selected as the entering variable.

		X1	X2	S1	S2	A1	A3		
BASIS	CJ	2	5	0	0	−M	−M	BI	Ratio
A1	−M	1	0	−1	0	1	0	4	4 (min)
S2	0	1	4	0	1	0	0	32	32
A3	−M	3	2	0	0	0	1	24	8
ZJ		−4M	−2M	M	0	−M	−M	−28M	
CJ − ZJ		2+4M	5+2M	−M	0	0	0		

FIGURE CD3.12*a* Iteration 1

		X1	X2	S1	S2	A1	A3		
BASIS	CJ	2	5	0	0	−M	−M	BI	Ratio
X1	2	1	0	−1	0	1	0	4 *	- -
S2	0	0	4	1	1	−1	0	28	32
A3	−M	0	2	3	0	−3	1	12	4 (min)
ZJ		2	−2M	−2−3M	0	2+3M	−M	8−12M	
CJ − ZJ		0	5+2M	2+3M	0	−2−4M	0		

FIGURE CD3.12*b* Iteration 2

		X1	X2	S1	S2	A1	A3		
BASIS	CJ	2	5	0	0	−M	−M	BI	Ratio
X1	2	1	2/3	0	0	0	1/3	8	12
S2	0	0	10/3	0	1	0	−1/3	24	72/10
S1	0	0	2/3	1	0	−1	1/3	4 *	6 (min)
ZJ		2	4/3	0	0	0	2/3	16	
CJ − ZJ		0	11/3	0	0	−M	−M−2/3		

FIGURE CD3.12*c* Iteration 3

		X1	X2	S1	S2	A1	A3	
BASIS	CJ	2	5	0	0	−M	−M	BI
X1	2	1	0	−1	0	1	0	4
S2	0	0	0	−5	1	5	−2	4 *
X2	5	0	1	3/2	0	−3/2	1/2	6
ZJ		2	5	11/2	0	−11/2	5/2	38
CJ − ZJ		0	0	−11/2	0	−M+11/2	−M−5/2	

FIGURE CD3.12*d* Iteration 4

The tableaus generated the following sequence of points:

Iteration	*Point*	*Classification*	*Reason*
1	X1 = 0, X2 = 0	Not feasible	A1 and A3 are both positive
2	X1 = 4, X2 = 0	Not feasible	A3 is positive
3	X1 = 8, X2 = 0	Feasible but not optimal	CJ-ZJ for X2 is positive
4	X1 = 4, X2 = 6	Feasible and optimal	All CJ-ZJ ≤ 0

This model is depicted graphically in Figure CD3.13. Since the third constraint is the equality 3X1 + 2X2 = 24, the feasible region is only the line segment of 3X1 + 2X2 = 24 between (8,0) and (4,6). The sequence of points generated by the simplex algorithm en route to the optimal solution is highlighted. Notice that the points corresponding to the first two iterations—(0,0) and (4,0)—each lie at the intersection of two constraint boundaries. These are basic solutions but *not* basic feasible solutions (extreme points), since they violate some constraints of the problem. The third basic solution generated (8,0) *is* an extreme point (basic feasible solution), but it is not optimal. The last point, (4,6), is the optimal extreme point (basic feasible solution) for the problem.

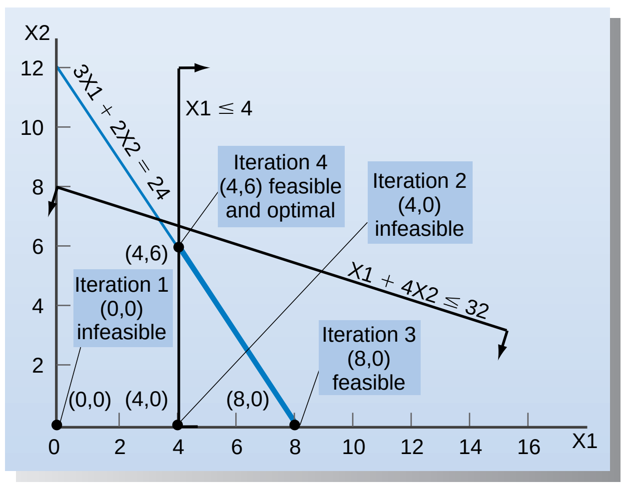

FIGURE CD3.13 Sequence of Points Generated by the Simplex Algorithm

VI Simplex Algorithm—Special Cases

In Chapter 3, we introduced graphically the concepts of minimization problems, unbounded linear programs, infeasible linear programs, alternative optimal solutions, and degeneracy. Here we discuss each in the context of the simplex algorithm.

MINIMIZATION PROBLEMS

Regardless of whether a problem has a maximization or a minimization objective function, the CJ-ZJ row shows the *net effect* on the objective function of increasing each variable by one unit. A *negative* CJ-ZJ value for a variable implies that increasing the variable *decreases* the value of the objective function. Since this is precisely the objective in minimization problems, the only modification to the simplex algorithm is to select the *most negative* CJ-ZJ value in Step 1, instead of the most positive value. The algorithm terminates, and the optimal solution is found when all the CJ-ZJ values are *nonpositive*. This is the approach we will use in our discussions.[4]

[4]*Note*: An alternative approach is first to multiply the minimization objective function by −1 and leave Step 1 as it is, finding the variable with the most positive CJ-ZJ. If we use this approach, then, when the optimal solution is found, we must multiply the objective function value by −1 to obtain the correct value for the minimization problem.

When artificial variables are added in minimization problems, they are assigned large *positive* objective function coefficients of +M to try to force them to zero values:

Modification For Minimization Problems

1. Add coefficients of +M to the objective function for artificial variables.
2. Change Step 1 of the simplex algorithm to find the most negative entry in the CJ-ZJ row; if there are no negative entries, *stop*; the current solution is optimal.

UNBOUNDED LINEAR PROGRAMS

In Step 2 of the simplex method, we determined the amount a current nonbasic variable can be increased before a current basic variable reaches zero. As we saw, the entry in a column of a nonbasic variable is the amount the corresponding basic variable will be *reduced* per unit increase in that nonbasic variable.

If there are no positive entries in the column, none of the current basic variables are reduced; they either increase (if the column entry is negative), or stay the same (if the column entry is zero). Thus, the entering variable can be increased without bound, generating an unbounded solution that yields an infinite profit, in a maximization problem, or infinite negative cost, in a minimization problem.

Unbounded Linear Programs

If all entries in the pivot column (in Step 2) are nonpositive, the linear program is unbounded.

INFEASIBLE LINEAR PROGRAMS

When artificial variables are added to a linear programming formulation, as long as an artificial variable has a positive value, the corresponding solution is infeasible. Adding the large negative objective function coefficients for artificial variables in maximization problems (−M) forces the simplex algorithm to try to make all the artificial variables assume zero values. If an artificial variable is positive, multiplying it by −M gives an objective function value that is effectively $-\infty$.

A*ny* feasible solution (one in which the values of all artificial variables are 0) gives an objective function value greater than $-\infty$. Thus, if the "optimal" tableau contains an artificial variable with a positive value, the problem must be infeasible. A parallel argument exists for minimization problems.

Infeasible Linear Programs

If an artificial variable remains positive in the "optimal tableau," the problem is infeasible.

ALTERNATE OPTIMAL SOLUTIONS

Suppose an optimal tableau has been found and some CJ-ZJ entry for a nonbasic variable is 0. Recall that a CJ-ZJ value provides the net change in the value of the objective function from increasing the corresponding variable by one unit. When a CJ-ZJ value for a nonbasic variable is 0, the net change in the objective function value generated by increasing this variable by any amount is 0. Thus, if we have already determined that a tableau is optimal because all the CJ-ZJs are ≤ 0 (≥ 0 for minimization problems), increasing this nonbasic variable from zero will generate alternate solutions with the same (optimal) value for the objective function.

Alternate Optimal Solutions

If the CJ-ZJ value for one or more nonbasic variables is 0 in the optimal tableau, alternate optimal solutions exist.

Recall that in Step 2 of the simplex method we determine the maximum increase for the entering variable (before some current basic variable reaches zero). Thus, if we perform Steps 2 and 3 of the simplex method, we get another *optimal* basic feasible solution (an adjacent optimal extreme point) with the same objective function value. In addition, *any weighted average of optimal solutions is also optimal*. For example, if the points (3, 6) and (5, 2) are optimal points, then .5(3, 6) + .5(5, 2) = (4, 4) is also an optimal point, as is .1(3, 6) + .9(5, 2) = (4.8, 2.4).

DEGENERATE LINEAR PROGRAMS

Once a problem has been put into standard form, all variables must be nonnegative (positive or zero). If a *basic* variable is zero (i.e., there is a "0" on the right-hand side of a feasible tableau), this solution is called a *degenerate solution*. The geometric interpretation of degeneracy—more than two constraint boundary lines intersect at the same point in a two-dimensional problem; more than three constraint boundary planes intersect at the same point in a three-dimensional problem; etc. was introduced in footnote 3 of Section 3.4.

Degenerate Solutions

If the value of a basic variable is 0 (the right-hand side value is 0) in a feasible tableau, the corresponding basic feasible solution is degenerate.

Algebraically, a degenerate solution can occur in two ways:

1. At *formulation*:
   The problem is formulated with a "0" on the right side.
2. *There is a tie for the minimum value of the ratio test*:
   Since this test determines which current basic variable reaches zero first as the entering variable is increased, a tie in the ratio test means that two or more current basic variables reach zero simultaneously.

A zero on the right-hand side of a simplex tableau is treated like any other number. When finding the ratios in Step 2 of the simplex algorithm, if the corresponding number in the entering column is a positive number (say 5), then the ratio for this row is 0/5 = 0; hence, this is the minimizing ratio determining the leaving variable for this iteration. On the other hand, if the corresponding number in the entering column is zero or negative, a ratio test is not performed on this row, and some other ratio is the minimizing one.

Suppose the pivot row has a zero on the right-hand side. Since the next tableau is generated by taking multiples of the pivot row and subtracting them from the other rows, the new right-hand side numbers of the next tableau are generated by subtracting multiples of zero; that is, the right-hand side of the next tableau is the same as that of the current tableau. This means that the solution will remain unchanged from the previous one. The only difference is that a different variable is identified as the basic variable at zero value.

This is theoretically troubling, for, as a result of this condition, the simplex method might get stuck "cycling" from tableau to tableau, generating the same degenerate point but with a different *basic* variable equaling zero each time. However, for nearly all known practical formulations, degeneracy does not present a problem. If there is a tie for the minimum value of the ratio test, by randomly selecting which of those variables is to be the nonbasic variable and which stays basic at zero value, the simplex algorithm is guaranteed not to cycle continuously.

Other methods (ϵ-perturbation and lexicographic ordering, to name two) can also be employed to guarantee that no basic feasible solution expressed with the same sequence of basic variables can ever be repeated. Since cycling rarely, if ever, occurs in practice, however, many computer codes simply ignore degeneracy.

VII SENSITIVITY ANALYSIS USING THE SIMPLEX METHOD

Commercial software packages perform sensitivity analyses by analyzing the optimal tableau of the original problem before changes are made. The question asked is, "How does the change in the original formulation affect the final tableau?" These effects are then analyzed to determine the conditions under which the tableau can still be viewed as optimal.

To be an optimal tableau, the tableau must satisfy three conditions:

1. The tableau must be in canonical form; that is, it must have a basic variable for each constraint.
2. All numbers on the right-hand side must be nonnegative.
3. All numbers in the CJ-ZJ row must be nonpositive for maximization problems (nonnegative for minimization problems).

Recall that the equations in the final tableau have been generated by simply performing a series of elementary row operations on the original set of equations.

Keeping these facts in mind, we will now illustrate how to use the simplex method to perform sensitivity analysis for the Galaxy Industries problem.

The original problem formulation after the addition of the slack variables is:

$$\begin{array}{ll} \text{MAX} & 8X1 + 5X2 \\ \text{ST} & \\ & 2X1 + X2 + S1 = 1200 \text{ (Plastic)} \\ & 3X1 + 4X2 + S2 = 2400 \text{ (Production Time)} \\ & X1 + X2 + S3 = 800 \text{ (Total Production)} \\ & X1 - X2 + S4 = 450 \text{ (Mix)} \\ & XJ, SJ \geq 0 \text{ for all J} \end{array}$$

The optimal tableau is shown in Figure CD3.14.

		X1	X2	S1	S2	S3	S4	
BASIS	CJ	8	5	0	0	0	0	BI
X2	5	0	1	−3/5	2/5	0	0	240
S4	0	0	0	−7/5	3/5	0	1	210
S3	0	0	0	−1/5	−1/5	1	0	80
X1	8	1	0	4/5	−1/5	0	0	480
ZJ		8	5	17/5	2/5	0	0	5040
CJ − ZJ		0	0	−17/5	−2/5	0	0	

FIGURE CD3.14 Optimal Tableau for Galaxy Industries Problem

RANGE OF OPTIMALITY FOR OBJECTIVE FUNCTION COEFFICIENTS (CJ)

To determine the range of optimality for an objective function coefficient of the variable XJ, we change the objective function coefficient for XJ in the CJ row of the final tableau to the generic value, CJ. If XJ is a basic variable, we must also change the value in the BASIS CJ column for XJ. We now consider what other changes result from this change in the final tableau.

Range of Optimality for Objective Function Coefficients of Basic Variables

To determine the range of optimality for the objective function coefficient for Space Rays (C1), we begin by changing its coefficient in the CJ row and the BASIS CJ column from 8 to C1. Since the equations in the final tableau are generated by row operations on the original set of equations, changing C1 (which is not part of an equation) *does not change the equations of the final tableau*. Changing the entry in the CJ column, however, means that the ZJ row must be recalculated in terms of C1. Accordingly, the entries in the CJ-ZJ row must also be revised and expressed in terms of C1. Making these changes to Figure CD3.10, we see that the final tableau for the Galaxy Industries problem yields the tableau shown in Figure CD3.15.

		X1	X2	S1	S2	S3	S4	
BASIS	CJ	C1	5	0	0	0	0	BI
X2	5	0	1	−3/5	2/5	0	0	240
S4	0	0	0	−7/5	3/5	0	1	210
S3	0	0	0	−1/5	−1/5	1	0	80
X1	C1	1	0	4/5	−1/5	0	0	480
ZJ		C1	5	−3+4/5C1	2−1/5C1	0	0	5040
CJ − ZJ		0	0	3−4/5C1	−2+1/5C1	0	0	

FIGURE CD3.15 Tableau For Calculating the Range of Optimality for C1

We can now ask, ''Under what conditions is this still an optimal tableau?'' Canonical form exists, and the entries on the right-hand side are still nonnegative. Thus, the tableau is still optimal as long as all entries in the CJ-ZJ row remain nonpositive; that is:

$$3 - 4/5\ C1 \leq 0 \tag{i}$$

and

$$-2 + 1/5\ C1 \leq 0 \tag{ii}$$

As long as C1 takes on values that satisfy both (i) and (ii), the tableau is optimal. We see that the following conditions exist:

$$3 - 4/5\ C1 \leq 0 \quad \text{implies} \quad -4/5\ C1 \leq -3 \quad \text{or} \quad C1 \geq 3.75 \tag{i}$$

$$-2 + 1/5\ C1 \leq 0 \quad \text{implies} \quad 1/5\ C1 \leq 2 \quad \text{or} \quad C1 \leq 10 \tag{ii}$$

Therefore, as long as $3.75 \leq C1 \leq 10$, the tableau is optimal. *This is the range of optimality for* C1. Some computer modules (such as WINQSB) print the range of optimality in this fashion; others (such as Excel and LINDO) express the result in terms of the maximum increase and maximum decrease in the original objective function coefficient for X1 (C1 = 8). In this case, these latter programs print for C1 a MAXIMUM INCREASE of 10 − 8 = 2, and a MAXIMUM DECREASE of 8 − 3.75 = 4.25.

In this example there were only two restrictions on C1—one mandated that C1 be greater than or equal to some value (3.75); the other required C1 to be less than or equal to another value (10). Had there been more restrictions, the range of optimality would have been determined by the most severe restrictions. For example, suppose another restriction on C1 is C1 ≥ 2. If C1 ≥ 3.75, then C1 ≥ 2 also, and the lower limit on the range of optimality remains at 3.75. If, however, an additional restriction states that C1 ≥ 4, then the new lower limit for the range of optimality becomes 4.

The method for calculating the range of optimality for objective function coefficients of basic variables is summarized as follows:

Calculating The Range of Optimality For Objective Function Coefficients of Basic Variables

1. Replace the numerical value of the objective function coefficient of XJ in the CJ row and the CJ column with the generic value "CJ."
2. Recalculate the ZJ row; then recalculate the CJ-ZJ row in terms of "CJ."
3. Determine the range of values for "CJ" so that all entries in the CJ-ZJ row remain nonpositive (nonnegative for minimization problems).

Range of Optimality for Objective Function Coefficients of Nonbasic Variables

In a maximization problem, any decrease in the objective function coefficient of a nonbasic variable makes the variable even less attractive. The value of the variable remains 0, and the optimal solution does not change, no matter how large or small the decrease. Thus, the lower limit for the range of optimality of a nonbasic variable is $-\infty$.

Recall that the reduced cost for a nonbasic variable (a nonpositive number in maximization problems) expresses the amount an objective function coefficient must decrease before that variable can become positive in the optimal solution. Since only basic variables can be positive, the "reduced cost" is the amount an objective function coefficient for a nonbasic variable must decrease before the variable can become *basic*.

Consider how the tableau is affected by increasing the objective function coefficient of a nonbasic variable. Its coefficient is changed to "CJ" in the objective function row, but since it is a nonbasic variable, *the* CJ *column remains unchanged*; hence, *the* ZJ *row remains unchanged as well*. Thus, the only change in the CJ-ZJ row is the CJ-ZJ entry for this one nonbasic variable.

Suppose the CJ value for a particular nonbasic variable is 6 and its ZJ value is 10. The CJ-ZJ entry is $6 - 10 = -4$. Thus the CJ value has to (increase by more than 4) decrease by more than -4 before the CJ-ZJ entry becomes positive and the variable becomes basic. This value, -4, is the reduced cost and is the CJ-ZJ entry in the optimal tableau. Its ZJ value, 10, is the *upper limit of the range of optimality* for the nonbasic variable.

For minimization problems, Step 1 of the simplex method is amended by selecting the variable with the most *negative* CJ-ZJ as the entering variable. Otherwise, the algorithm remains the same. Using this approach, we observe that the CJ-ZJ value for a nonbasic variable still provides the variable's reduced cost. Now, however, the upper bound of the range of optimality for a nonbasic variable is $+\infty$, and the lower bound is its ZJ value.

Reduced Costs/Range of Optimality For Nonbasic Variables

1. The *reduced cost* for a nonbasic variable is its CJ-ZJ value.
2. Range of optimality for a nonbasic variable:
   Maximization problems: ($-\infty$, its ZJ value)
   Minimization problems: (its ZJ value, ∞)

RANGE OF FEASIBILITY FOR RIGHT-HAND SIDE COEFFICIENTS (BI)

The range of feasibility for a right-hand side coefficient is the range of values for the coefficient such that the same constraints determine the optimal solution. In terms of the simplex approach, the range of feasibility is determined by the range of values for the right-hand side coefficient such that the same set of basic variables make up the optimal solution.

To illustrate how to determine the range of feasibility for a resource, we shall calculate the range of feasibility for plastic (B1) in the Galaxy Industries problem. Suppose a change (increase or decrease) occurs in the availability of plastic from its current value of 1200 pounds. Let us denote this change by the mathematical symbol "ΔB1" (the change in B1). Thus, the right-hand side values of the original problem formulation change from

1200		1200 + ΔB1
2400	to	2400
800		800
450		450

If we express the original right-hand side in two columns—a column listing the original right-hand side values before the change, and a variable column, "ΔB1"—then the right-hand side can be expressed as

Constant	*ΔB1*
1200	1
2400	0
800	0
450	0

The original tableau is depicted in Figure CD3.16.

BASIS	CJ	X1 8	X2 5	S1 0	S2 0	S3 0	S4 0	BI	Δ B1
S1	0	2	1	1	0	0	0	1200	1
S2	0	3	4	0	1	0	0	2400	0
S3	0	1	1	0	0	1	0	800	0
S4	0	1	−1	0	0	0	1	450	0
ZJ		0	0	0	0	0	0	0	
CJ − ZJ		8	5	0	0	0	0		

FIGURE CD3.16 Original Tableau with ΔB1 Added

Before the addition of the ΔB1 column, we generated the optimal tableau for the problem (given in Figure CD3.10) by performing a sequence of elementary row operations that transformed the original X1 column in Figure CD3.2 (2, 3, 1, 1) to the X1 column of Figure CD3.10 (0, 0, 0, 1), the X2 column from (1, 4, 1, −1) to (1, 0, 0, 0), the S1 column from (1, 0, 0, 0) to (−3/5, −7/5, −1/5, 4/5), and so on. The original right-hand side column (BI) was transformed from (1200, 2400, 800, 450) to (240, 210, 80, 480).

Performing the same sequence of row operations on Figure CD3.16 results in the same final tableau, except now there is an additional "ΔB1" column. The question then becomes, "Into what will this column (which began as (1, 0, 0, 0)) be transformed?" To answer this question, we need look no further than the S1 column, which also began as a (1, 0, 0, 0) column and was transformed into (−3/5, −7/5, −1/5, 4/5). Thus, these same coefficients are in the "ΔB1" column of the final tableau, as shown in Figure CD3.17.

		X1	X2	S1	S2	S3	S4		
BASIS	CJ	8	5	0	0	0	0	BI	Δ B1
X2	5	0	1	−3/5	2/5	0	0	240	−3/5
S4	0	0	0	−7/5	3/5	0	1	210	−7/5
S3	0	0	0	−1/5	−1/5	1	0	80	−1/5
X1	8	1	0	4/5	−1/5	0	0	480	4/5
ZJ		8	5	17/5	2/5	0	0	5040	17/5
CJ − ZJ		0	0	−17/5	−2/5	0	0		

FIGURE CD3.17 Tableau For Calculating the Range of Feasibility for BI

The question remains, "Is Figure CD3.17 the optimal tableau?" The answer is "yes" as long as all the right-hand side values in the final tableau are nonnegative. Recall that the coefficients in the "ΔB1" column multiply the variable, "ΔB1". Thus, the actual right-hand side values in the final tableau are:

$$240 - 3/5\Delta B1$$
$$210 - 7/5\Delta B1$$
$$80 - 1/5\Delta B1$$
$$480 + 4/5\Delta B1$$

All of these quantities must be "≥ 0." The calculations that provide the limits on "ΔB1" for which this condition is satisfied are as follows:

$$240 - 3/5\Delta B1 \geq 0 \quad \text{implies} \quad -3/5\Delta B1 \geq -240 \quad \text{or} \quad \Delta B1 \leq 400$$
$$210 - 7/5\Delta B1 \geq 0 \quad \text{implies} \quad -7/5\Delta B1 \geq -210 \quad \text{or} \quad \Delta B1 \leq 150$$
$$80 - 1/5\Delta B1 \geq 0 \quad \text{implies} \quad -1/5\Delta B1 \geq -80 \quad \text{or} \quad \Delta B1 \leq 400$$
$$480 + 4/5\Delta B1 \geq 0 \quad \text{implies} \quad 4/5\Delta B1 \geq -480 \quad \text{or} \quad \Delta B1 \geq -600$$

From these restrictions, ΔB1 must not be less than −600; since ΔB1 must not exceed either 150 or 400, it can not be greater than 150. Thus, the range of feasibility expressed in terms of "the change to B1" is:

$$-600 \leq \Delta B1 \leq 150$$

Some computer programs (such as Excel and LINDO) present the range of feasibility by stating that 600 is the MAXIMUM DECREASE, and 150 is the MAXIMUM INCREASE to B1. Others, such as WINQSB, print the range of feasibility in absolute terms, not changes. Since the original value of B1 was 1200, according to the above analysis, B1 can be decreased by 600 to 1200 − 600 = 600, or increased by 150 to 1200 + 150 = 1350. Thus, expressed in terms of the "value of B1," the range of feasibility is:

$$600 \leq B1 \leq 1350$$

We use similar reasoning to determine the range of feasibility for the other right-hand side values. For example, to determine the range of feasibility for B2, the original "ΔB2" column is (0, 1, 0, 0). Thus, the entries in the final tableau for the "ΔB2" column are the entries in the S2 column, or (2/5, 3/5, −1/5, −1/5). These multiply "ΔB2" and are added to (240, 210, 80, 480), respectively, in the final tableau to determine the range of feasibility for B2. Similarly, the product of (0, 0, 1, 0) times "ΔB3" is added to (240, 210, 80, 480) in the final tableau to determine the range of feasibility for B3; and the product of (0, 1, 0, 0) times "ΔB4" is added to (240, 210, 80, 480) in the final tableau to determine the range of feasibility for B4.

Since the original "ΔBI" column is a unit column, the entries in the "ΔBI" column in the final tableau are the same as those in the corresponding *slack* variable column, SI, assuming that the I-th constraint is a "≤" constraint. If the I-th constraint is an "="

or a "≥" constraint, the corresponding unit column in the original tableau is associated with an artificial variable. Hence, the entries in the "ΔBI" column of the final tableau are the entries in the corresponding *artificial* variable column, AI.

SHADOW PRICES

The ZJ value in the "ΔBI" column is the amount the objective function changes per unit change in the I-th resource; this is the definition of its *shadow price*. The ZJ value in the "ΔBI" column, however, is the same as the ZJ value for the corresponding slack variable (SI) or artificial variable (AI). Thus, *the shadow price of the I-th resource is the ZJ value of the corresponding slack variable*, SI, *or artificial variable*, AI. As Figure CD3.10 illustrates, the shadow prices for plastic, production minutes, total production, and product mix for Galaxy Industries are 17/5, 2/5, 0, and 0, respectively.

Calculating The Range Of Feasibility For BI and the Shadow Price For the I-th Resource

Range of Feasibility

1. Express the right-hand side values in the final tableau by adding "ΔBI" times the I-th slack or artificial column to the right-hand side values of the final tableau.
2. Determine the limits on "ΔBI" that keep all of these entries ≥ 0. This is the range of feasibility expressed in terms of "ΔBI."
3. Add the original value of BI to both the upper and lower limits for "ΔBI" to express the range of feasibility in terms of BI.

Shadow Price for the I-th Resource

Constraint Type	***Shadow Price***
"≤"	ZJ value of the slack variable for the row I
"="	ZJ value of the artificial variable for row I
"≥"	ZJ value of the artificial variable for row I

VIII The dual simplex method

The *dual simplex method* can be used to solve any linear program; in fact, it is an effective approach to solving minimization problems with positive objective function coefficients and all "greater than or equal to" constraints. Its most common use, however, is to generate a new optimal solution when a change in a right-hand side coefficient extends beyond its range of feasibility or when a new constraint is added to a problem after an optimal solution has been found.

Recall that, for a tableau to be optimal, all three of the following conditions must hold:

1. The tableau must be in canonical form.
2. The right-hand side must be nonnegative.
3. All CJ-ZJ values must be nonpositive (nonnegative for minimization problems).

At every iteration of the simplex algorithm, conditions 1 and 2 are satisfied; the optimal solution is found when condition 3 is met. By contrast, the dual simplex algorithm is used when conditions 1 and 3 are met; the optimal solution is found when condition 2 is satisfied as well.

USING THE DUAL SIMPLEX METHOD WHEN THE RANGE OF FEASIBILITY IS VIOLATED

To illustrate the use of the dual simplex method, let us return to the Galaxy Industries problem. In Section VII we showed how changes in the availability of plastic (B1) yielded the following right-hand side values in the final tableau, expressed in terms of ΔB1 (ΔB1 is the change in B1 from its original value of 1200):

$$240 - 3/5\ \Delta B1$$
$$210 - 7/5\ \Delta B1$$
$$80 - 1/5\ \Delta B1$$
$$480 + 4/5\ \Delta B1$$

The value of the objective function expressed in terms of ΔB1 is 5040 + 17/5 ΔB1. Thus, had B1 been increased from 1200 to 1300 (ΔB1 = 100), the optimal solution would have been given by the following modified right-hand side values:

$$X2 = 240 - 3/5\ (100) = 180$$
$$S4 = 210 - 7/5\ (100) = 70$$
$$S3 = 80 - 1/5\ (100) = 60$$
$$X1 = 480 + 4/5\ (100) = 560$$

and the optimal objective function value would have been 5400 + 17/5(100) = 5740. Had B1 increased by ΔB1 = 200 to 1400 (which is outside the range of feasibility), however, the new right-hand side would have been:

$$X2 = 240 - 3/5\ (200) = 120$$
$$S4 = 210 - 7/5\ (200) = -70$$
$$S3 = 80 - 1/5\ (200) = 40$$
$$X1 = 480 + 4/5\ (200) = 640$$

and the tableau would be that shown in Figure CD3.18. This tableau is in canonical form, and all CJ-ZJ ≤ 0; however, a negative number appears on the right-hand side. The dual simplex can then be used to generate a new optimal solution.

The Dual Simplex Method

1. Select the *leaving row*—the row with the most negative right-hand side number. If there are no negative numbers on the right-hand side, STOP—the tableau is optimal.
2. Determine the entering column by finding the minimizing ratio between the numbers in the CJ-ZJ row and *negative* numbers in the leaving row. If there are no negative numbers in the leaving row, STOP—the problem is infeasible.

   (For minimization problems, the CJ-ZJ row entries are nonnegative and the entering column is derived by finding the maximizing ratio—minimizing in terms of absolute value—between the CJ-ZJ row and negative numbers in the leaving row.)
3. Conduct the standard pivot operation; the pivot element is the intersection of the leaving row and the entering column.

Then Go back to STEP 1.

BASIS	CJ	X1	X2	S1	S2	S3	S4	BI
		8	5	0	0	0	0	
X2	5	0	1	−3/5	2/5	0	0	120
S4	0	0	0	−7/5	3/5	0	1	−70
S3	0	0	0	−1/5	−1/5	1	0	40
X1	8	1	0	4/5	−1/5	0	0	640
ZJ		8	5	17/5	2/5	0	0	5720
CJ − ZJ		0	0	−17/5	−2/5	0	0	

FIGURE CD3.18 Amended Tableau After Changing BI to 1400

Step 1 generates a new positive entry on the right-hand side at the next iteration, while the ratio test in Step 2 ensures that the optimality criteria (all CJ-ZJ values ≤ 0 for maximization problems, or all CJ-ZJ values ≥ 0 for minimization problems) are maintained at the next iteration.

Performing Step 1 of the dual simplex method on the tableau above, we find only one negative right-hand side entry, −70. Hence, the second row is the leaving row, and S4 is the leaving variable. The ratio test (Step 2) is applied only to *negative* entries in the second row. Since there is only one negative entry in this row (−7/5), it is the pivot element. Performing the standard pivot operation using −7/5 as the pivot element gives us Figure CD3.19.

BASIS	CJ	X1	X2	S1	S2	S3	S4	BI
		8	5	0	0	0	0	
X2	5	0	1	0	1/7	0	−3/7	150
S1	0	0	0	1	−3/7	0	−5/7	50
S3	0	0	0	0	−2/7	1	−1/7	50
X1	8	1	0	0	1/7	0	4/7	600
ZJ		8	5	0	13/7	0	17/7	5550
CJ − ZJ		0	0	0	−13/7	0	−17/7	

FIGURE CD3.19 Tableau After an Iteration of the Dual Simplex Algorithm

Since all entries on the right-hand side of this tableau are nonnegative, this tableau is optimal. The new optimal solution is X1 = 600, X2 = 150, S1 = 50, S2 = 0, S3 = 50, S4 = 0, and the new optimal value of the objective function is $5550.

USING THE DUAL SIMPLEX METHOD WHEN CONSTRAINTS ARE ADDED AFTER THE PROBLEM HAS BEEN SOLVED

Suppose that, after the original problem at Galaxy Industries is solved, giving the solution X1 = 480, X2 = 240, management imposes a new constraint:

$$X1 + 1/2\ X2 \leq 500$$

Substituting the current solution into the left side of this constraint gives 480 + 1/2(240) = 600; thus, this constraint is violated by the current optimal solution. At this point, we add this new constraint, including its slack variable, S5, to the bottom of the current tableau. In doing so, however, we destroy the canonical form of the tableau because there is no "0" in this row for the column of the basic variable, X1, and the basic variable, X2.

As indicated in Figure CD3.20, we can reestablish a "0" in the X1 column of the new constraint by subtracting "1" times the row for which X1 is the basic variable (row 4) from this new constraint. Similarly, we can reestablish a "0" in the X2 column of the new constraint by subtracting "1/2" times the row for which X2 is the basic variable (row 1) from this new constraint. The results are presented in Figure CD3.20.

		X1	X2	S1	S2	S3	S4	S5	
BASIS	CJ	8	5	0	0	0	0	0	BI
X2	5	0	1	−3/5	2/5	0	0	0	240
S4	0	0	0	−7/5	3/5	0	1	0	210
S3	0	0	0	−1/5	−1/5	1	0	0	80
X1	8	1	0	4/5	−1/5	0	0	0	480
S5	0	1	1/2	0	0	0	0	1	500
ZJ		8	5	17/5	2/5	0	0	0	5040
CJ − ZJ		0	0	−17/5	−2/5	0	0	0	

Not canonical form

Subtract 1 times row 4 and 1/2 times row 1 from row 5

New Constraints in Canonical Form

		X1	X2	S1	S2	S3	S4	S5	
BASIS	CJ	8	5	0	0	0	0	0	BI
X2	5	0	1	−3/5	2/5	0	0	0	240
S4	0	0	0	−7/5	3/5	0	1	0	210
S3	0	0	0	−1/5	−1/5	1	0	0	80
X1	8	1	0	4/5	−1/5	0	0	0	480
S5	0	0	0	−1/2	0	0	0	1	−100
ZJ		8	5	17/5	2/5	0	0	0	5040
CJ − ZJ		0	0	−17/5	−2/5	0	0	0	

FIGURE CD3.20 Adding the Constraint $X1 + \frac{1}{2} \times 2 \leq 500$

This new tableau is now in canonical form and has all nonpositive entries in the CJ-ZJ row; however, the right-hand side of the last constraint contains a negative entry (−100). These are precisely the conditions given above for performing the dual simplex method.

At this iteration, S5 is the leaving variable. Since there is only one negative number in the leaving row (in the S1 column), S1 is the entering variable and the −1/2 in that row is the pivot element. Performing the standard pivot operations generates the results shown in Figure CD3.21. All right-hand side entries are now nonnegative; thus, this is the optimal tableau. The new optimal solution is X1 = 320, X2 = 360, S1 = 200, S2 = 0, S3 = 120, S4 = 490, S5 = 0, and the optimal value of the objective function is $4360.

BASIS	CJ	X1	X2	S1	S2	S3	S4	S5	BI
		8	5	0	0	0	0	0	
X2	5	0	1	0	2/5	0	0	−6/5	360
S4	0	0	0	0	3/5	0	1	−14/5	490
S3	0	0	0	0	−1/5	1	0	−2/7	120
X1	8	1	0	0	−1/5	0	0	8/5	320
S1	0	0	0	1	0	0	0	−2	200
ZJ		8	5	0	2/5	0	0	34/5	4360
CJ − ZJ		0	0	0	−2/5	0	0	−34/5	

FIGURE CD3.21 Tableau After Pivoting

IX Duality and Tableaus

In Supplement CD2, we introduced the concept of *duality*. There we showed that any linear programming problem called the primal has an associated linear programming problem called the dual. If the primal is a maximization problem, the dual is a minimization problem. For Galaxy Industries, the primal-dual pair of programs is:

Primal Problem

$$
\begin{array}{ll}
\text{MAX} & 8X1 + 5X2 \\
\text{ST} & 2X1 + X2 \leq 1200 \\
 & 3X1 + 4X2 \leq 2400 \\
 & X1 + X2 \leq 800 \\
 & X1 - X2 \leq 450 \\
 & X1, X2 \geq 0
\end{array}
$$

Dual Problem

$$
\begin{array}{ll}
\text{MIN} & 1200Y1 + 2400Y2 + 800Y3 + 450Y4 \\
\text{ST} & 2Y1 + 3Y2 + Y3 + Y4 \geq 8 \\
 & Y1 + 4Y2 + Y3 - Y4 \geq 5 \\
 & Y1, Y2, Y3, Y4 \geq 0
\end{array}
$$

We can use the optimal tableau of either problem to find the optimal solutions for both the primal and dual problems. The locations of both the primal and dual variables in the optimal tableau for the primal problem are as follows:

Location of the Optimal Primal and Dual Variables in the Optimal Tableau for the Primal Problem

Value of	***Location in Tableau***
All primal variables	Basic variables—RHS values Nonbasic variables—0
Dual variables	ZJ values of the corresponding slack variables (for associated "≤" constraints) and ZJ values of the corresponding artificial variables (for associated "=" and "≥"

Figure CD3.22 details these values for the Galaxy Industries problem.

BASIS	CJ	X1	X2	S1	S2	S3	S4	BI
		8	5	0	0	0	0	
X2	5	0	1	$-3/5$	$2/5$	0	0	240
S4	0	0	0	$-7/5$	$3/5$	0	1	210
S3	0	0	0	$-1/5$	$-1/5$	1	0	80
X1	8	1	0	$4/5$	$-1/5$	0	0	480
ZJ		8	5	$17/5$	$2/5$	0	0	5040
CJ − ZJ		0	0	$-17/5$	$-2/5$	0	0	

Negative of the slack/surplus variables of the dual

Value of the dual variables

Value of the primal basic variables (including slack/surplus variables)

FIGURE CD3.22 Optimal Primal and Dual Solutions from the Optimal Tableau of the Primal

Problems

1. STANDARD FORM/CANONICAL FORM Given the following linear programming formulation:

$$\begin{aligned} \text{MAX}\quad & 4X1 + 5X2 + 2X3 \\ \text{ST}\quad & \\ & 2X1 - X2 - 2X3 \geq 8 \\ & X1 + 2X2 + X3 = 16 \\ & X1 - 2X2 + X3 \leq 12 \\ & X1 \geq 0,\ X2 \text{ unrestricted},\ X3 \leq 0 \end{aligned}$$

a. Convert the formulation to standard form.

b. Add artificial variables where appropriate and give the first canonical form.

c. Suppose the right-hand side of the first constraint were −8 instead of 8, and the right-hand side of the third constraint were −12 instead of 12. Rewrite the original constraints so that all right-hand side values are nonnegative by multiplying the first and third constraints by −1 (do not forget to change the inequality sign when multiplying by a negative number) and convert this formulation to standard form.

2. STANDARD FORM/SIMPLEX (MAXIMIZATION) Given the following problem:

$$\begin{aligned} \text{MAX}\quad & 5X1 + 4X2 + 6X3 \\ \text{ST}\quad & \\ & 2X1 + X2 + X3 \leq 40 \\ & 3X1 + 4X2 + X3 \leq 100 \\ & X1 - 4X2 + 4X3 \leq 24 \\ & \text{ALL } XJ \geq 0 \end{aligned}$$

a. Write the problem in standard form. Is this also canonical form?

b. Solve the problem using the simplex algorithm.

3. STANDARD FORM/SIMPLEX (MINIMIZATION) Given the following problem:

$$\begin{aligned} \text{MIN}\quad & 2X1 + 3X2 - 4X3 - X4 \\ \text{ST}\quad & \\ & X1 - 2X2 + 2X3 + X4 \leq 10 \\ & 2X1 + 2X2 + 3X3 + X4 \leq 30 \\ & X1 - X2 + 4X3 - X4 \leq 40 \\ & \text{ALL } XJ \geq 0 \end{aligned}$$

a. Write the problem in standard form. Is this also canonical form?

b. Solve the problem using the simplex algorithm.

4. SIMPLEX ALGORITHM/GRAPH Given the following problem:

$$\begin{aligned} &\text{MAX} \quad 3X1 + 4X2 \\ &\text{ST} \\ &\qquad 2X1 + X2 \le 12 \\ &\qquad X1 + 2X2 \le 9 \\ &\qquad X1 + 4X2 \le 16 \\ &\qquad X1, X2 \ge 0 \end{aligned}$$

a. Solve the problem by the simplex algorithm.

b. Graph the problem and show which point is generated at each step of the algorithm.

c. In Section III we pointed out that choosing any variable with a positive CJ-ZJ improves the objective function value from one iteration to the next and that selecting the one with the most positive CJ-ZJ as the entering variable does not always provide the optimal solution in the fewest number of iterations. In fact, Step 1 could simply be, ''*Choose the first variable encountered that has a positive* CJ-ZJ *value as the entering variable*.'' Using this rule, show how the optimal solution for this problem is reached in a smaller number of iterations. Show the sequence of points on the graph generated by each tableau using this rule.

5. SIMPLEX ALGORITHM/ROUND-OFF ERROR Given the following problem:

$$\begin{aligned} &\text{MAX} \quad 5X1 + 8X2 + 10X3 \\ &\text{ST} \\ &\qquad 2X1 + 3X2 + 6X3 \le 18 \\ &\qquad X1 + 4X2 + 2X3 \le 12 \\ &\qquad 3X1 - X2 + 5X3 \le 14 \\ &\qquad XJ \ge 0 \text{ for all J} \end{aligned}$$

a. Solve the problem by the simplex algorithm expressing all numbers at every iteration as fractions.

b. Re-solve the problem by the simplex algorithm expressing all numbers at every iteration in two-place decimal format. Compare your answer with part (a).

c. The computer keeps numbers as decimals written out to many decimal places; still, there is round-off error. Use WINQSB, Excel, or LINDO to solve this problem. Compare your answers.

6. ARTIFICIAL VARIABLES (MAXIMIZATION) Given the following problem:

$$\begin{aligned} &\text{MAX} \quad 3X1 + 4X2 + 5X3 \\ &\text{ST} \\ &\qquad 2X1 + X2 \ge 8 \\ &\qquad 2X2 + X3 \le 6 \\ &\qquad 3X1 + 2X3 = 18 \\ &\qquad XJ \ge 0 \text{ for all J} \end{aligned}$$

a. Write the problem in standard form.

b. Write the problem in canonical form.

c. Use the simplex algorithm to solve for the optimal solution.

7. ARTIFICIAL VARIABLES (MINIMIZATION) Given the following problem:

$$\begin{aligned} &\text{MIN} \quad 2X1 + 10X2 + 5X3 \\ &\text{ST} \\ &\qquad X1 + 4X2 + 2X3 \ge 20 \\ &\qquad 4X1 + 8X2 - X3 = 40 \\ &\qquad 3X1 + X2 + 2X3 \le 50 \\ &\qquad XJ \ge 0 \text{ for all J} \end{aligned}$$

a. Write the problem in standard form.

b. Write the problem in canonical form.

c. Use the simplex algorithm to solve for the optimal solution.

8. ARTIFICIAL VARIABLES/GRAPH Given the following problem:

$$\begin{array}{ll} \text{MAX} & 5X1 + 8X2 \\ \text{ST} & \\ & 2X1 + 3X2 \geq 9 \\ & X1 + X2 \leq 6 \\ & 5X1 + 2X2 = 18 \\ & X1, X2 \geq 0 \end{array}$$

a. Solve the problem graphically for the optimal solution.

b. Solve the problem using the simplex algorithm.

c. On the graph, illustrate the solution (whether feasible or infeasible) for each tableau. Which points are basic solutions? Which points are basic feasible solutions?

9. ARTIFICIAL VARIABLES/TWO-PHASE METHOD When solving for the optimal solution to a problem in which artificial variables have been added, there are two objectives: (1) to make sure that all artificial variables turn out to be 0; and (2) to determine an optimal solution. The two-phase method, which treats these two objectives separately, is an alternative to the BIG M method of assigning each artificial variable an objective function coefficient of −M (+M for minimization problems).

Consider Problem 6 above. Artificial variables A1 and A3 should have been added to the first and third constraints, respectively, to obtain the first canonical form.

a. PHASE 1—Ignore the original objective function and impose an alternative objective of minimizing the sum of the artificial variables; that is, MIN A1 + A3. Obviously, since A1 and A3 must be ≥ 0, a minimum value for this objective function of 0 can be attained only if both A1 and A3 are 0. This accomplishes the first objective. (*Note: If* A1 *and* A3 *cannot both be made* 0, *the problem is infeasible.*) Impose the objective function MIN A1 + A3 and use the simplex method to solve for the optimal solution for the Phase 1 problem for Problem 6.

b. PHASE 2—The canonical form that gave the optimal solution to Phase 1 corresponds to a basic feasible solution. Now re-impose the original objective function: MAX 3X1 + 4X2 + 5X3. Calculate the ZJ row and the CJ-ZJ row and perform the normal simplex algorithm to find the optimal solution. (*Note: Either drop the artificial variable columns altogether from your tableau, or simply do not choose an artificial variable as an entering variable at any time. The only reason to keep the artificial variable columns is for sensitivity analysis information.*)

10. INFEASIBILITY Using the simplex algorithm, show that the following problem is infeasible:

$$\begin{array}{ll} \text{MAX} & 4X1 + 6X2 + 5X3 \\ \text{ST} & \\ & 2X1 + 3X2 + 8X3 \leq 10 \\ & X1 + 2X2 + X3 \geq 4 \\ & X1 - X2 = 6 \\ & XJ \geq 0 \text{ for all J} \end{array}$$

11. INFEASIBILITY/GRAPH Given the following problem:

$$\begin{array}{ll} \text{MAX} & 2X1 + 3X2 \\ \text{ST} & \\ & X1 \geq 6 \\ & X1 + X2 \leq 10 \\ & 3X1 + 4X2 = 36 \\ & X1, X2 \geq 0 \end{array}$$

a. Solve the problem graphically and show that there is no feasible solution.

b. Using the simplex algorithm, show that the problem is infeasible.

c. On the graph, illustrate the solution associated with each tableau. Do these points correspond to basic solutions? Do they correspond to basic feasible solutions?

12. UNBOUNDED PROBLEM (MAXIMIZATION) Given the following problem:

$$\begin{aligned} \text{MAX} \quad & 4X1 + X2 + 2X3 \\ \text{ST} \quad & \\ & 2X1 + X2 - 2X3 \leq 10 \\ & X1 - 2X2 - X3 \leq 20 \\ & 3X1 - 2X2 + X3 \leq 30 \\ & XJ \geq 0 \text{ for all J} \end{aligned}$$

a. Solve the problem by the simplex algorithm and show that the problem is unbounded.

b. Using the tableau from which you determined that the problem is unbounded, how much will each of the following increase for every unit that X2 is increased:
   i. the value of the objective function
   ii. the values of all other variables

   These relationships are collectively known as the *unbounded solution*.

13. UNBOUNDED FEASIBLE REGION/BOUNDED SOLUTION Since Problem 12 yielded an unbounded solution, the constraint set must form an unbounded feasible region. Using the same constraints as in Problem 12, show that the problem with the following objective function gives an optimal solution:

$$\text{MAX } 2X1 - 3X2 + X3$$

14. UNBOUNDED PROBLEM (MINIMIZATION) Use the simplex algorithm to show that the following problem is unbounded:

$$\begin{aligned} \text{MIN} \quad & 6X1 - 3X2 + 2X3 \\ \text{ST} \quad & \\ & X1 + X2 - X3 \leq 10 \\ & 2X1 - X2 - 2X3 \leq 12 \\ & 4X1 - 2X2 + X3 \leq 20 \\ & XJ \geq 0 \text{ for all J} \end{aligned}$$

15. UNBOUNDED PROBLEM/GRAPH Given the following problem:

$$\begin{aligned} \text{MAX} \quad & 4X1 + 3X2 \\ \text{ST} \quad & \\ & 2X1 - X2 \geq 8 \\ & -X1 + 4X2 \leq 10 \\ & X1, X2 \geq 0 \end{aligned}$$

a. Show graphically that the linear program is unbounded.

b. Show that the problem is unbounded using the simplex method.

c. On the graph, illustrate the solution corresponding to each tableau. From your last tableau, show the points that are generated when S1 is increased from 0.

16. ALTERNATE OPTIMAL SOLUTIONS/GRAPH Given the following problem:

$$\begin{aligned} \text{MAX} \quad & 9X1 + 6X2 \\ \text{ST} \quad & \\ & 3X1 + 2X2 \leq 18 \\ & X1 + 2X2 \leq 12 \\ & X1 - 2X2 \leq 4 \\ & X1, X2 \geq 0 \end{aligned}$$

a. Solve the problem graphically and show that it has alternate optimal solutions.

b. Using simplex algorithm, show that this problem has alternate optimal solutions.

c. On the graph, illustrate the solution corresponding to each tableau. How can the set of alternate optimal solutions be generated from the simplex tableau?

17. ALTERNATE OPTIMAL SOLUTIONS Given the following problem:

$$\begin{aligned} \text{MAX} \quad & 15X1 + 26X2 + 10X3 \\ \text{ST} \quad & \\ & 2X1 + 4X2 + X3 \leq 70 \\ & 3X1 + 2X2 + 4X3 \leq 168 \\ & X1 + 2X2 + X3 \leq 63 \\ & XJ \geq 0 \text{ for all J} \end{aligned}$$

a. Show that the problem has alternate optimal solutions.

b. Generate two optimal basic feasible solutions.

c. Weight the values of the variables found in your two solutions in part (b) by 50% each (i.e., multiply the values of all variables in each solution by .5 and add the results). Show that this weighted solution is also feasible and gives the same value for the objective function.

18. DEGENERACY Show that the following linear program has a degenerate optimal solution:

$$\begin{aligned} &\text{MAX} \quad 11X1 + 12X2 + 8X3 \\ &\text{ST} \\ &\quad 2X1 + 5X2 + 4X3 \leq 16 \\ &\quad 3X1 + 2X2 + X3 \leq 4 \\ &\quad X1 - X2 + X3 \leq 6 \\ &\quad XJ \geq 0 \text{ for all J} \end{aligned}$$

19. DEGENERACY DISAPPEARS If a problem gives a degenerate solution at one iteration, this does not mean that the optimal solution is degenerate. Consider the following problem in which the initial basic feasible solution is degenerate. Show that the optimal solution is *not* a degenerate solution.

$$\begin{aligned} &\text{MAX} \quad 4X1 + 5X2 + 8X3 \\ &\text{ST} \\ &\quad X1 + 2X2 - X3 \leq 0 \\ &\quad X1 + X2 + X3 \leq 7 \\ &\quad 2X1 + 3X2 + 5X3 \leq 21 \\ &\quad XJ \geq 0 \text{ for all J} \end{aligned}$$

20. DUAL SIMPLEX METHOD Given the following problem:

$$\begin{aligned} &\text{MIN} \quad 3X1 + 2X2 + 7X3 + 4X4 \\ &\text{ST} \\ &\quad X1 + 2X2 + 3X3 + 2X4 \geq 20 \\ &\quad 2X1 + X2 - X3 + 3X4 \geq 16 \\ &\quad XJ \geq 0 \text{ for all J} \end{aligned}$$

a. Solve the problem using the dual simplex algorithm.

b. Solve the problem using the simplex algorithm.

21. SENSITIVITY ANALYSIS (MAXIMIZATION) Consider Problem 2 of this supplement.

a. Determine the range of optimality for each decision variable.

b. Determine the reduced cost for each decision variable.

c. Determine the values of the shadow prices for each constraint.

d. Determine the range of feasibility for the right-hand side of each constraint.

22. SENSITIVITY ANALYSIS/GRAPH (MAXIMIZATION) Consider Problem 4 of this supplement.

a. Determine the range of optimality for each decision variable.

b. Determine the reduced cost for each decision variable.

c. Determine the values of the shadow prices for each constraint.

d. Determine the range of feasibility for the right-hand side of each constraint.

e. Verify your results for parts (a) through (d) using a graphical analysis to obtain these results.

23. SENSITIVITY ANALYSIS (MINIMIZATION) Consider Problem 3 of this supplement.

a. Determine the range of optimality for each decision variable.

b. Determine the reduced cost for each decision variable.

c. Determine the values of the shadow prices for each constraint.

d. Determine the range of feasibility for the right-hand side of each constraint.

24. SENSITIVITY ANALYSIS/ARTIFICIAL VARIABLES (MAXIMIZATION) Consider Problem 8 of this supplement.

a. Determine the range of optimality for each decision variable.

b. Determine the reduced cost for each decision variable.

c. Determine the values of the shadow prices for each constraint.
d. Determine the range of feasibility for the right hand-side of each constraint.

25. SENSITIVITY ANALYSIS (MINIMIZATION) Consider Problem 20 of this supplement.

a. Determine the range of optimality for each decision variable.
b. Determine the reduced cost for each decision variable.
c. Determine the values of the shadow prices for each constraint.
d. Determine the range of feasibility for the right-hand side of each constraint.

26. POST-OPTIMALITY ANALYSES (MAXIMIZATION) Consider Problem 2 of this supplement. Use post-optimality analyses to determine the new optimal solution in each of the following cases:

a. The following constraint is added: $2X1 + 3X2 + X3 \leq 80$
b. The following constraint is added: $2X1 + 3X2 + X3 \leq 60$
c. The following constraint is added: $X1 \geq 10$
d. A variable is added with an objective function coefficient of +8 and coefficients in the three functional constraints of 2, 4, and 1, respectively.
e. A variable is added with an objective function coefficient of +12 and coefficients in the three functional constraints of 2, 4, and 1, respectively.

27. POST-OPTIMALITY ANALYSES (MAXIMIZATION) Consider Problem 6 of this supplement. Use post-optimality analyses to determine the new optimal solution in each of the following cases:

a. The following constraint is added: $2X1 - 4X2 + X3 \leq 5$
b. The following constraint is added: $2X1 - 4X2 + X3 \geq 5$
c. The following constraint is added: $X2 \geq 4$
d. A variable is added with an objective function coefficient of +2 and coefficients in the constraints of 1, 1, and 1, respectively.
e. A variable is added with an objective function coefficient of +4 and coefficients in the constraints of 1, 1, and 1, respectively.

28. POST-OPTIMALITY ANALYSES (MINIMIZATION) Consider Problem 20 of this supplement. Use post-optimality analyses to determine the new optimal solution in each of the following cases:

a. The following constraint is added: $X1 + 3X2 + X4 \geq 20$
b. The following constraint is added: $X1 + 3X2 + X4 \geq 30$
c. The following constraint is added: $X2 \leq 5$
d. The following constraint is added: $X1 + X2 + X3 + X4 \leq 9$
e. A variable is added with an objective function coefficient of +9 and coefficients in the constraints of 3 and 1, respectively.
f. A variable is added with an objective function coefficient of +2 and coefficients in the constraints of 3 and 1, respectively.

29. DUALITY Consider Problem 2 of this supplement.

a. Formulate the dual to this problem.
b. Using the optimal tableau to the primal problem, determine the optimal solution to the dual problem, including the values of the dual surplus variables.
c. Solve the dual problem using the simplex algorithm and verify your answers to part (b).
d. Solve the dual problem using the dual simplex algorithm. Compare your results to those obtained solving the primal problem using the simplex algorithm.

30. DUALITY Consider Problem 6 of this supplement.

a. Formulate the dual to this problem.
b. Using the optimal tableau to the primal problem, determine the optimal solution to the dual problem, including the values of all slack and surplus variables.